# 1-König-Egerváry Graphs

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**Abstract**

Let $\alpha(G)$ denote the cardinality of a maximum independent set, while $\mu(G)$ be the size of a maximum matching in $G=(V,E)$. It is known that if $\alpha(G)+\mu(G)=|V|$, then $G$ is a *König-Egerváry graph [2, 9]*. If $\alpha(G)+\mu(G)=|V|-1$, then $G$ is an 1-*König-Egerváry graph*. If $G$ is not a König-Egerváry graph, and there exists a vertex $v \in V$ (an edge $e \in E$) such that $G-v$ $(G-e)$ is König-Egerváry, then $G$ is called a vertex (an edge) almost König-Egerváry graph (respectively).

In this paper, we characterize all these types of almost König-Egerváry graphs and present interrelationships between them.

**Keywords:** maximum independent set, matching, König-Egerváry graph.

## 1 Introduction

Throughout this paper $G=(V,E)$ is a finite, undirected, loopless graph without multiple edges, with vertex set $V=V(G)$ of cardinality $|V(G)|=n(G)$, and edge set $E=E(G)$ of size $|E(G)|=m(G)$.

If $X \subset V$, then $G[X]$ is the subgraph of $G$ induced by $X$. By $G-v$ we mean the subgraph $G[V-\{v\}]$, for $v \in V$. The *neighborhood* of a vertex $v \in V$ is the set $N(v)=\{w : w \in V \text{ and } vw \in E\}$. The *neighborhood* of $A \subseteq V$ is $N(A)=\{v \in V : N(v) \cap A \neq \emptyset\}$.

A set $S \subseteq V$ is *independent* if no two vertices belonging to $S$ are adjacent. The *independence number* $\alpha(G)$ is the size of a largest independent set (i.e., of a *maximum independent set*) of $G$.

A set $S \subseteq V(G)$ is *independent* if no two vertices from $S$ are adjacent, and by $\mathrm{Ind}(G)$ we mean the family of all the independent sets of $G$. An independent set of maximum size is a *maximum independent set* of $G$, and $\alpha(G)=\max\{|S| : S \in \mathrm{Ind}(G)\}$. Let $\Omega(G)$ be the family of all maximum independent sets, and $\mathrm{core}(G)=\bigcap\{S : S \in \Omega(G)\}$ [4].

A *matching* in a graph $G=(V,E)$ is a set of edges $M \subseteq E$ such that no two edges of $M$ share a common vertex. A matching of maximum cardinality $\mu(G)$ is a *maximum matching*, and a *perfect matching* is one saturating all vertices of $G$. Given a matching $M$ in $G$, a vertex $v \in V$ is called $M$-saturated if there exists an edge $e \in M$ incident with $v$.

An edge $e \in E(G)$ is *$\mu$-critical* provided $\mu(G-e) < \mu(G)$. A vertex $v \in V(G)$ is *$\mu$-critical (essential)* provided $\mu(G-v) < \mu(G)$, i.e., $v$ is $M$-saturated by every maximum matching $M$ of $G$.

It is known that $\lfloor n(G)/2 \rfloor + 1 \leq \alpha(G) + \mu(G) \leq n(G) \leq \alpha(G) + 2\mu(G)$ hold for every graph $G$ [1]. If $\alpha(G) + \mu(G) = n(G)$, then $G$ is called a König-Egerváry graph [2, 9].

**Theorem 1.1** *[7] For a graph $G$, the following properties are equivalent:*
(i) *$G$ is a König-Egerváry graph;*
(ii) *each maximum matching of $G$ matches $V(G) - S$ into $S$, for some $S \in \Omega(G)$;*
(iii) *each maximum matching of $G$ matches $V(G) - S$ into $S$, for every $S \in \Omega(G)$.*

**Definition 1.2** *If $\alpha(G)+\mu(G) = n(G)-1$, then $G$ is called an 1-König-Egerváry graph.*

For instance, both $G_1$ and $G_2$ from Figure 1 are 1-König-Egerváry graphs.

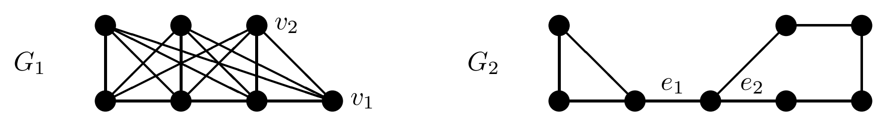


Figure 1: $G_1 - v_1$, $G_2 - e_2$ are König-Egerváry graphs, while $G_1 - v_2$ and $G_2 - e_1$ are not König-Egerváry graphs.

**Definition 1.3** *A graph $G$ is called:*
(i) *a vertex almost König-Egerváry graph if $G$ is not König-Egerváry, but there is a vertex $v \in V(G)$, such that $G - v$ is a König-Egerváry graph [3];*
(ii) *an edge almost König-Egerváry graph if $G$ is not a König-Egerváry graph, but there exists an edge $e \in E(G)$, such that $G - e$ is a König-Egerváry graph.*

For example, the graph $G_1$ from Figure 1 is vertex almost König-Egerváry but not edge almost König-Egerváry, while $G_2$ is edge almost König-Egerváry but not vertex almost König-Egerváry.

**Lemma 1.4** *[3] A graph $G$ is a vertex almost König-Egerváry graph if and only if there is a vertex $v \in V(G)$ such that $G - v$ is a König-Egerváry graph, $\alpha(G-v) = \alpha(G)$, and $\mu(G-v) = \mu(G)$.*

Clearly, every odd cycle $C_{2k+1}$ is a vertex almost König-Egerváry graph, an edge almost König-Egerváry graph and an almost König-Egerváry graph.

**Lemma 1.5** *For a graph $G$, the following assertions hold:*
(i) $\alpha(G) \leq \alpha(G-e) \leq \alpha(G) + 1$ *for each* $e \in E(G)$;
(ii) $\alpha(G) - 1 \leq \alpha(G-v) \leq \alpha(G)$ *for every* $v \in V(G)$;
(iii) $\mu(G) - 1 \leq \mu(G-a) \leq \mu(G)$ *for each* $a \in V(G) \cup E(G)$.

Notice that: each edge of $C_{2q+1}$ is $\alpha$-critical and non-$\mu$-critical; each vertex of $C_{2q+1}$ is neither $\alpha$-critical and nor $\mu$-critical; each edge of $C_{2q}$ is neither $\alpha$-critical and nor $\mu$-critical; each vertex of $C_{2q}$ is $\mu$-critical, but non-$\alpha$-critical.

Using Lemma 1.5, one can easily state the following.

**Corollary 1.6** *The equality $\alpha(G-e)+\mu(G-e)=\alpha(G)+\mu(G)$ holds if and only if either the edge $e\in E(G)$ is both $\alpha$-critical and $\mu$-critical or the edge $e\in E(G)$ is both non-$\alpha$-critical and non-$\mu$-critical.*

For instance, consider the graphs from Figure 2:

- $\alpha(G_1-a)+\mu(G_1-a)=\alpha(G_1)+\mu(G_1)=6$, since $a$ is both $\alpha$-critical and $\mu$-critical;
- $\alpha(G_2-u_1)+\mu(G_2-u_1)=\alpha(G_2)+\mu(G_2)+1=7$, since $u_1$ is $\alpha$-critical and non-$\mu$-critical;
- $\alpha(G_2-u_2)+\mu(G_2-u_2)=\alpha(G_2)+\mu(G_2)=6$, since $u_2$ is both non-$\alpha$-critical and non-$\mu$-critical;
- $\alpha(G_3-b)+\mu(G_3-b)=\alpha(G_3)+\mu(G_3)-1=4$, since $b$ is non-$\alpha$-critical and $\mu$-critical.

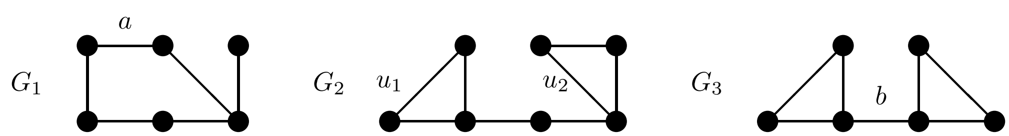


Figure 2: Only $G_1$ is a König-Egerváry graph.

**Proposition 1.7** *[5] In a König-Egerváry graph $\alpha$-critical edges are also $\mu$-critical, and they coincide in bipartite graphs.*

In this paper, we characterize 1-König-Egerváry graphs, almost vertex (edge) König-Egerváry graphs, and present interrelationships between them.

## 2 Results

**Definition 2.1** *A set $A\subseteq V(G)$ is supportive if either*
*(i) there is a vertex $v\in V(G)-A$ and a matching from $V(G)-A-v$ into $A$, or*
*(ii) there is an edge $xy\in E(G-A)$ and a matching from $V(G)-A-x-y$ into $A$.*

For instance, consider the graphs in Figure 3: the set $A=\{a_1,a_2\}$ is a supportive (maximum independent) set in $G_1$, and $B=\{b_1,b_2\}$ is a supportive set in $G_1$.

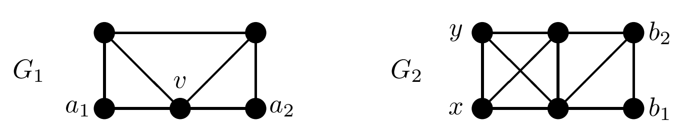


Figure 3: Supportive sets.

The following finding gives a structural characterization of 1-König-Egerváry graphs, similarly to the one for König-Egerváry graphs (Theorem 1.1).

**Theorem 2.2** *Let $G$ be a non-König-Egerváry graph. Then the following assertions are equivalent:*

*(i) $G$ is a 1-König-Egerváry graph;*

*(ii) there exists a supportive maximum independent set in $G$;*

*(iii) every maximum independent set of $G$ is supportive.*

**Proof.** *(i) $\Rightarrow$ (iii)* Assume that $G$ is 1-König-Egerváry, $S$ is an arbitrary maximum independent set, and $M$ is a maximum matching. Hence,

$$\mu(G) = n(G) - \alpha(G) - 1 = n(G) - |S| - 1.$$

Let $M$ contain $b_1$ edges connecting $S$ and $V(G) - S$, while $b_2$ edges connecting vertices from $V(G) - S$. Thus $\mu(G) = b_1 + b_2$. Hence,

$$n(G) - \alpha(G) = n(G) - |S| \geq b_1 + 2b_2 = \mu(G) + b_2.$$

Therefore,

$$1 = n(G) - \alpha(G) - \mu(G) \geq b_2.$$

*Case 1.* $b_2 = 0$. Since $|M| = \mu(G) = n(G) - \alpha(G) - 1 = |V(G) - S| - 1$, we infer that $M$ saturates all the vertices from $V(G) - S$, except one, say $v \in V(G) - S$, and then $G - v$ is a König-Egerváry graph.

*Case 2.* $b_2 = 1$. Since $|M| - 1 = \mu(G) - 1 = n(G) - \alpha(G) - 2 = |V(G) - S| - 2$, we conclude that $M$ saturates all the vertices from $V(G) - S$, and $M$ contains exactly one edge that joins two vertices from $V(G) - S$.

Thus $S$ is supportive.

*(iii) $\Rightarrow$ (ii)* Clear.

*(ii) $\Rightarrow$ (i)* Let $S$ be a supportive maximum independent set. Now, by the definition of a supportive set and the fact that $S$ is a maximum independent set, we get $\alpha(G)+\mu(G) = n(G) - 1$. ■

**Corollary 2.3** *A non-König-Egerváry graph $G$ is 1-König-Egerváry if and only if either there is a vertex $v \in V(G)$ such that $G - v$ is a König-Egerváry graph or there is an edge $xy \in E(G)$ such that $G - x - y$ is a König-Egerváry graph.*

**Proof.** Suppose that $G$ is 1-König-Egerváry. By Theorems 1.1 and 2.2, either there exists a vertex $v$ such that $G - v$ is König-Egerváry or there is an edge $xy$ such that $G - x - y$ is König-Egerváry.

Conversely, assume that $G - v$ is a König-Egerváry graph, for some $v \in V(G)$. Since $G$ is not a König-Egerváry graph, we get that

$$n(G) - 1 = \alpha(G - v) + \mu(G - v) \leq \alpha(G) + \mu(G) \leq n(G) - 1,$$

which means that $\alpha(G) + \mu(G) = n(G) - 1$.

Now, let $xy \in E(G)$ be such that $G - x - y$ is a König-Egerváry graph. Hence,

$$n(G) - 2 = \alpha(G - x - y) + \mu(G - x - y) \leq \alpha(G) + \mu(G) \leq n(G) - 1.$$

It is clear that $\alpha(G) - \alpha(G - x - y) \geq 0$. On the other hand, $\mu(G) - \mu(G - x - y) > 0$, because $xy \in E(G)$. Thus $\alpha(G) + \mu(G) = n(G) - 1$, as required. ■

For instance, the graphs from Figure 4 are 1-König-Egerváry, as both $G_1 - x - y$ and $G_2 - v$ are König-Egerváry graphs.

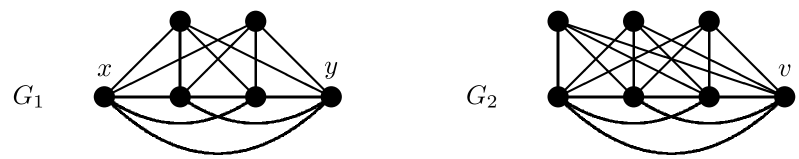


Figure 4: Both $G_1$ and $G_2$ are almost König-Egerváry graphs.

**Theorem 2.4** *If $G$ is an edge almost König-Egerváry graph or a vertex almost König-Egerváry graph, then $G$ is a 1-König-Egerváry graph as well.*

**Proof.** Let $e \in E(G)$ be such that $G - e$ is a König-Egerváry graph. Since $G$ is not a König-Egerváry graph, and clearly, $\alpha(G) \leq \alpha(G - e) \leq \alpha(G) + 1$ and $\mu(G) \geq \mu(G - e)$, we obtain

$$n(G) - 1 = n(G - e) - 1 = \alpha(G - e) - 1 + \mu(G - e) \leq \alpha(G) + \mu(G) < n(G).$$

Therefore, $\alpha(G) + \mu(G) = n(G) - 1$, i.e., $G$ is a 1-König-Egerváry graph.

Let $v \in V(G)$ be such that $G - v$ is a König-Egerváry graph. Since $G$ is not a König-Egerváry graph, we deduce that

$$n(G) - 1 = n(G - v) = \alpha(G - v) + \mu(G - v) \leq \alpha(G) + \mu(G) < n(G),$$

which implies that $\alpha(G) + \mu(G) = n(G) - 1$, i.e., $G$ is a 1-König-Egerváry graph. ■

Notice that there exist 1-König-Egerváry graphs that are neither edge almost König-Egerváry graphs, nor vertex almost König-Egerváry graphs, e.g., $pK_1 + K_{p+1}$, $pK_1 + K_{p+2}$.

In continuation of Lemma 1.4 we proceed with the following.

**Theorem 2.5** (i) *A graph $G$ is vertex almost König-Egerváry if and only if it is 1-König-Egerváry and some $v \in V(G)$ is neither $\alpha$-critical nor $\mu$-critical.*

(ii) *A graph $G$ is edge almost König-Egerváry if and only if it is 1-König-Egerváry and some $e \in E(G)$ is $\alpha$-critical and non-$\mu$-critical.*

**Proof.** Since $G$ is not a König-Egerváry graph, we know that $\alpha(G) + \mu(G) < n(G)$.

*(i)* If $G$ is vertex almost König-Egerváry, then it is also 1-König-Egerváry, by Theorem 2.4.

There exists $v \in V(G)$ such that $G - v$ is a König-Egerváry graph. According to Lemma 1.5*(ii)* and *(iii)*, we get

$$n(G) - 1 = \alpha(G - v) + \mu(G - v) \leq \alpha(G) + \mu(G) < n(G),$$

which implies that $\alpha(G - v) = \alpha(G)$ and $\mu(G - v) = \mu(G)$, and these mean that $v \in V(G)$ is neither $\alpha$-critical nor $\mu$-critical.

The converse is clear, because $\alpha(G - v) + \mu(G - v) = \alpha(G) + \mu(G) = n(G) - 1$.

*(ii)* If $G$ is edge almost König-Egerváry, then it is also 1-König-Egerváry, by Theorem 2.4.

There exists $xy \in E(G)$ such that $G - xy$ is a König-Egerváry graph. According to Lemma 1.5*(i)* and *(iii)*, we get

$$n(G) = \alpha(G - xy) + \mu(G - xy) \leq \alpha(G) + 1 + \mu(G) < n(G) + 1,$$

which implies that $\alpha(G - xy) = \alpha(G) + 1$ and $\mu(G - xy) = \mu(G)$, i.e., the edge $xy$ is $\alpha$-critical and non-$\mu$-critical.

Conversely, we have that $\alpha(G-xy) = \alpha(G)+1$, $\mu(G-xy) = \mu(G)$, and $\alpha(G)+\mu(G) = n(G) - 1$, which ensures that

$$\alpha(G - xy) + \mu(G - xy) = \alpha(G) + \mu(G) + 1 = n(G),$$

and this means that $G$ is edge almost König-Egerváry. ■

A graph $G$ is *almost bipartite* if it has a unique odd cycle [8].

**Lemma 2.6** *[8] If $G$ is an almost bipartite graph, then $n(G)-1 \leq \alpha(G)+\mu(G) \leq n(G)$.*

Hence, each almost bipartite graph is either König-Egerváry or 1-König-Egerváry. Thus a non-König-Egerváry almost bipartite graph is 1-König-Egerváry. Moreover, it is both a vertex and an edge almost König-Egerváry graph.

**Corollary 2.7** *If $G$ is an almost bipartite graph, then the following assertions are equivalent:*
(i) *$G$ is a 1-König-Egerváry graph;*
(ii) *$G$ is a vertex almost König-Egerváry graph;*
(iii) *$G$ is an edge almost König-Egerváry graph.*

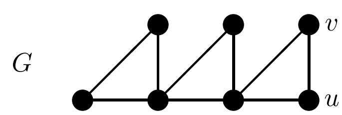


Figure 5: An almost König-Egerváry graph.

The graph $G$ from Figure 5 is not almost bipartite. By Theorem 2.5, $G$ is both a vertex and an edge almost König-Egerváry graph (since the vertex $v$ is neither $\alpha$-critical nor $\mu$-critical, while the edge $uv$ is $\alpha$-critical and non-$\mu$-critical).

Clearly, a unicyclic graph with an odd cycle is almost bipartite. Thus Corollary 2.7 goes for non-bipartite unicyclic graps as well [6].

# 3 Conclusions

A graph $G$ is a *critical edge almost König-Egerváry graph*, if $G$ is not a König-Egerváry graph, but $G - e$ is a König-Egerváry graph for every $e \in E(G)$. For instance, every $C_{2k+1}$ is a critical edge almost König-Egerváry graph.

**Lemma 3.1** *If $G$ is a critical edge almost König-Egerváry graph, then every edge is $\alpha$-critical.*

**Proof.** For each $e \in E(G)$, we have

$$\alpha(G) + \mu(G) \leq \alpha(G-e) + \mu(G) \leq n(G) = \alpha(G-e) + \mu(G-e),$$

which implies that $\mu(G) \leq \mu(G-e)$. Therefore, $\mu(G-e) = \mu(G)$ and further,

$$\alpha(G) + \mu(G) < n(G) = \alpha(G-e) + \mu(G)$$

ensures that $e \in E(G)$ must be $\alpha$-critical. ■

The converse is not true; e.g., $K_n, n \geq 5$.

A graph $G$ is a *critical vertex almost König-Egerváry graph*, if $G$ is not a König-Egerváry graph, but $G - v$ is a König-Egerváry graph for every $v \in V(G)$. For instance, every $C_{2k+1}$ is a critical vertex almost König-Egerváry graph.

**Lemma 3.2** *If $G$ is a critical vertex almost König-Egerváry graph, then* $\text{core}(G) = \emptyset$ *and $G$ has no perfect matching.*

**Proof.** For every $v \in V(G)$ we have

$$\alpha(G) + \mu(G) = n(G) - 1 = \alpha(G-v) + \mu(G-v),$$

which, by Lemma 1.5, means that $\alpha(G) = \alpha(G-v)$ and $\mu(G) = \mu(G-v)$. Consequently, $\text{core}(G) = \emptyset$ and $G$ has no perfect matching. ■

The converse is not true; e.g., $K_{2n+1}, n \geq 2$.

**Problem 3.3** *Characterize critical edge almost König-Egerváry graphs and critical vertex almost König-Egerváry graphs.*

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