

# Inequalities Connecting the Annihilation and Independence Numbers

Ohr Kadrawi  
 Department of Mathematics  
 Ariel University  
 Ariel 4070000, Israel  
 orka@ariel.ac.il

Vadim E. Levit  
 Department of Mathematics  
 Ariel University  
 Ariel 4070000, Israel  
 levitv@ariel.ac.il

August 4, 2023

## Abstract

Given a graph  $G$ , the number of its vertices is represented by  $n(G)$ , while the number of its edges is denoted as  $m(G)$ . An *independent set* in a graph is a set of vertices where no two vertices are adjacent to each other and the size of the maximum independent set is denoted by  $\alpha(G)$ . A *matching* in a graph refers to a set of edges where no two edges share a common vertex and the maximum matching size is denoted by  $\mu(G)$ . If  $\alpha(G) + \mu(G) = n(G)$ , then the graph  $G$  is called a *König–Egerváry graph*.

Considering a graph  $G$  with a degree sequence  $d_1 \leq d_2 \leq \dots \leq d_n$ , the *annihilation number*  $a(G)$  is defined as the largest integer  $k$  such that the sum of the first  $k$  degrees in the sequence is less than or equal to  $m(G)$  (Pepper, 2004).

It is a known fact that  $\alpha(G)$  is less than or equal to  $a(G)$  for any graph  $G$ . Our goal is to estimate the difference between these two parameters. Specifically, we prove a series of inequalities, including  $a(G) - \alpha(G) \leq \frac{\mu(G)-1}{2}$  for trees,  $a(G) - \alpha(G) \leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$  for bipartite graphs and  $a(G) - \alpha(G) \leq \mu(G) - 2$  for König–Egerváry graphs. Furthermore, we demonstrate that these inequalities serve as tight upper bounds for the difference between the annihilation and independence numbers, regardless of the assigned value for  $\mu(G)$ .

**Keywords**— annihilation number, independence number, tree, bipartite graph, König–Egerváry graph.

# 1 Introduction

In this paper, we consider a finite, undirected graph  $G = (V, E)$  without loops or multiple edges. The graph has a vertex set denoted by  $V(G)$  with a cardinality of  $|V(G)| = n(G)$ , and an edge set denoted by  $E(G)$  with a cardinality of  $|E(G)| = m(G)$ .

A subset  $S \subseteq V(G)$  is considered *independent* if no two vertices in  $S$  are adjacent. The collection of all independent sets of  $G$  is denoted as  $Ind(G)$ . A *maximum independent set* of  $G$  is an independent set with the largest possible size. The *independence number* of  $G$  is denoted as  $\alpha(G)$  and is defined as the maximum cardinality among all sets  $S \in Ind(G)$ .

A *matching* in a graph  $G$  refers to a set of edges  $M \subseteq E(G)$  where no two edges in  $M$  share a common vertex. A *maximum matching*, denoted as  $\mu(G)$ , is a matching with the largest possible cardinality. The cardinality of a maximum matching is called the *matching number* of the graph.

A graph is a *bipartite* if and only if it does not contain odd cycles. Clearly, every subgraph of a bipartite graph is also bipartite.

For any graph  $G$ , it is well-known, as stated in [3], that the following inequalities hold:  $\alpha(G) + \mu(G) \leq n(G) \leq \alpha(G) + 2\mu(G)$ . If a graph  $G$  satisfies the condition  $\alpha(G) + \mu(G) = n(G)$ , it is referred to as a *König–Egerváry* graph, as mentioned in [8, 22]. Notably, all bipartite graphs and trees belong to the class of König–Egerváry graphs.

Consider the *degree sequence* of a graph  $G$  given by  $d_1 \leq d_2 \leq \dots \leq d_a \leq d_{a+1} \leq \dots \leq d_n$ . The *annihilation number* of  $G$ , denoted as  $a(G)$ , was introduced by Pepper [20, 21]. It is defined as the largest integer  $k$  such that the sum of the first  $k$  terms in the degree sequence is no greater than half the sum of all the degrees in that sequence. In other words,  $a(G)$  is the maximum value of  $k$  satisfying  $\sum_{i=1}^k d_i \leq m(G)$ .

Let  $A \subseteq V(G)$  be a subset of vertices. The notation  $deg(A)$  denotes the sum of the degrees of the vertices in  $A$ , i.e.,  $\sum_{v \in A} d(v)$ . An *annihilating set* refers to any subset  $A \subseteq V(G)$  that satisfies the condition  $deg(A) \leq m(G)$ . It is evident that every independent set is also an annihilating set.

An annihilating set  $A$  is considered *maximal* if for every vertex  $x \in V(G) - A$ , adding  $x$  to  $A$  results in  $deg(A \cup x) > m(G)$ . On the other hand, an annihilating set  $A$  is called *maximum* if its cardinality is equal to the annihilation number of the graph, i.e.,  $|A| = a(G)$ , as stated in [21].

For instance, consider a graph  $G = K_{p,q} = (A, B, E)$  where  $p > q$ . In this case,  $A$  is a maximum annihilating set, while  $B$  is a maximal annihilating set.

**Theorem 1.1.** [17, 21] *For every graph  $G$ ,  $a(G) \geq \max\{\alpha(G), \frac{n(G)}{2}\}$ .*

Extensive research has been conducted to explore the relationship between the annihilation number and various parameters of a graph [1, 2, 4, 5, 6, 7, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21].

**Lemma 1.2.** *For every graph  $G$ ,  $a(G) - \alpha(G) \leq n(G) - 1$ .*

*Proof.* The annihilation number  $a(G)$  is bounded by the number of vertices  $n(G)$  since  $a(G)$  corresponds to an index in the degree sequence. The equality  $a(G) = n(G)$  can only occur when the graph has no edges, i.e.,  $m(G) = 0$ . It is worth noting that every single vertex in a graph forms an independent set, so for any graph  $G$ , we have  $\alpha(G) \geq 1$ . Therefore, this lemma holds for every graph.  $\square$

**Lemma 1.3.** *If  $G$  is a König–Egerváry graph, then  $0 \leq a(G) - \alpha(G) \leq \mu(G)$ . Moreover, if  $m(G) > 0$ , then  $a(G) - \alpha(G) < \mu(G)$ .*

*Proof.* Pepper [21] demonstrated that the inequality  $0 \leq a(G) - \alpha(G)$  holds. As for the right side, we know that  $a(G) \leq n(G) = \alpha(G) + \mu(G)$ . Consequently, we have  $a(G) - \alpha(G) \leq \mu(G)$ .

If  $m(G) > 0$ , then  $a(G) < n(G)$ , which completes the proof.  $\square$

**Lemma 1.4.** *For every König–Egerváry graph  $G$ ,  $\mu(G) \leq \alpha(G)$ .*

*Proof.* Notice that  $\mu(G)$  is less than or equal to half the value of  $n(G)$  because every edge  $e \in M(G)$  consists of two vertices and no two edges in  $M(G)$  share common vertices. Additionally,  $\alpha(G) = n(G) - \mu(G)$ , so we can conclude that:

$$\mu(G) \leq \frac{n(G)}{2} \leq \alpha(G).$$

$\square$

In this paper, we focus on determining tight upper bounds for the difference between the annihilation number and the independence number for different types of graphs. Specifically, in Section 2, we examine trees, in Section 3, we analyze bipartite graphs, and in Section 4, we investigate König–Egerváry graphs.

## 2 The tight upper bound for trees

The initial inequality concerning the difference between the annihilation number and the independence number of trees can be stated as follows:

$$a(T) - \alpha(T) \leq \mu(T).$$

This inequality is derived from Lemma 1.3, considering that a tree is a König–Egerváry graph.

To establish a tighter bound, we define the annihilation decomposition.

**Definition 2.1.** *An annihilation decomposition of a graph  $G$  is a partition  $\langle A, B \rangle$  of its vertex set to a maximum annihilation set and its complement. In what follows, we define  $k\langle A, B \rangle$  as the number of edges between vertices of  $A$  and  $B$ .*

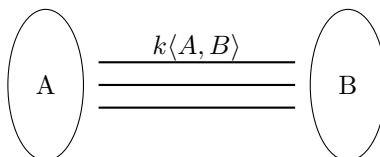


Figure 1: An annihilation decomposition of a graph.

It is important to note that in an annihilation decomposition,  $|A| = a(G)$  and  $|B| = n(G) - a(G)$ .

**Lemma 2.2.** *In an annihilation decomposition  $\langle A, B \rangle$  of a graph  $G$ , it holds that  $m(G[A]) \leq m(G[B])$ .*

*Proof.* It is known that the sum of all degrees in a graph is equal to twice the number of edges, i.e.,  $2m(G)$ . According to the definition of the annihilation number,  $a$  represents the maximum index in the degree sequence such that the sum of the degrees of vertices  $v_i$  for  $i \leq a$  is less than or equal to  $m(G)$ . These vertices are included in the set  $A$ .

On the other hand, the sum of the degrees of vertices in set  $B$  is greater than or equal to  $m(G)$ . Consequently, there are more (or an equal number of) edges in  $G[B]$  compared to  $G[A]$ .  $\square$

**Lemma 2.3.** *Let  $G$  be a bipartite graph with an annihilation decomposition  $\langle A, B \rangle$ . Then  $a(G) - \alpha(G) \leq m(G[A])$ .*

*Proof.* First, notice that for any graph  $G$ , the maximum matching number  $\mu(G)$  is always less than or equal to the number of edges  $m(G)$  because  $M \subseteq E(G)$ .

Let us now prove the inequality  $a(G) - \alpha(G) \leq m(G[A])$  by contradiction.

Suppose that  $m(G[A]) < a(G) - \alpha(G)$ . Then,

$$a(G) - \alpha(G[A]) = n(G[A]) - \alpha(G[A]) = \mu(G[A]) \leq m(G[A]) < a(G) - \alpha(G).$$

This implies that  $\alpha(G[A]) > \alpha(G)$ , which leads to a contradiction. Hence, the assumption that  $m(G[A]) < a(G) - \alpha(G)$  must be false, and we conclude that  $a(G) - \alpha(G) \leq m(G[A])$ .  $\square$

In the next part of this section, our objective is to establish the tight upper bound specifically for trees. To accomplish this, we can utilize Lemma 2.2 and Lemma 2.3, adapting them for trees by substituting  $T$  in place of  $G$ .

Let  $T$  be a tree with annihilation decomposition  $\langle A, B \rangle$ . The sum of the degrees in  $A$  is equal to twice the number of edges in  $A$  (since each edge has two endpoints in  $A$ ), which is denoted as  $2m(T[A])$  plus the edges between  $A$  and  $B$  (counted once), which is equal to  $k\langle A, B \rangle$ . Using Lemma 2.2, we can obtain the following inequality:

$$2m(T[A]) + k\langle A, B \rangle \leq m(T) \leq 2m(T[B]) + k\langle A, B \rangle.$$

By employing Lemma 2.3, we can establish the inequality  $a(T) - \alpha(T) \leq m(T[A])$  on the left side of the aforementioned inequality. Consequently, we can express it as:

$$2(a(T) - \alpha(T)) + k\langle A, B \rangle \leq 2(m(T[A])) + k\langle A, B \rangle.$$

On the right side, since  $T[B]$  is a subgraph of a tree, it can be regarded as a forest, and the maximum number of edges it can have is  $n(T[B]) - 1$ . Thus, we can write:

$$2m(T[B]) + k\langle A, B \rangle \leq 2(n(T[B]) - 1) + k\langle A, B \rangle.$$

By combining the above inequalities, we obtain the following:

**Theorem 2.4.** *For every tree  $T$ ,  $a(T) - \alpha(T) \leq \frac{\mu(T)}{2} - \frac{1}{2}$ .*

*Proof.* Based on the aforementioned calculations, we can represent these inequalities in the following manner:

$$\text{Deg}(A) = 2m(T[A]) + k\langle A, B \rangle \leq m(T) \leq 2m(T[B]) + k\langle A, B \rangle = \text{Deg}(B)$$

$$\begin{aligned} 2(a(T) - \alpha(T)) + k\langle A, B \rangle &\leq 2m(T[A]) + k\langle A, B \rangle \leq m(T) \leq \\ &\leq 2m(T[B]) + k\langle A, B \rangle \leq 2(n(T[B]) - 1) + k\langle A, B \rangle. \end{aligned}$$

Therefore,

$$2(a(T) - \alpha(T)) + k\langle A, B \rangle \leq 2(n(T[B]) - 1) + k\langle A, B \rangle$$

$$\begin{aligned} a(T) - \alpha(T) &\leq n(T[B]) - 1 \\ a(T) - \alpha(T) &\leq n(T) - a(T) - 1 \\ a(T) - \alpha(T) &\leq \alpha(T) + \mu(T) - a(T) - 1 \\ 2(a(T) - \alpha(T)) &\leq \mu(T) - 1 \\ a(T) - \alpha(T) &\leq \frac{\mu(T)}{2} - \frac{1}{2}. \end{aligned}$$

□

**Theorem 2.5.** *The inequality  $a(T) - \alpha(T) \leq \frac{\mu(T)}{2} - \frac{1}{2}$  represents the tight bound. In other words, there exist a tree for which  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$ .*

*Proof.* Let us take a star graph with 6 vertices. In this case, we have  $a(T) = 5$ ,  $\alpha(T) = 5$ , and  $\mu(T) = 1$ . As demonstrated, the equality holds for this tree.

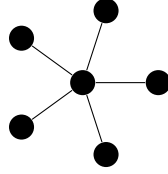


Figure 2: A star graph with 6 vertices.

□

Notice that we can expand and characterize the aforementioned graph as follows.

**Lemma 2.6.** *There exists an infinite family of graphs that satisfy the equality  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$  with  $\mu(T) = 1$ .*

*Proof.* Let us take a star graph of size  $n \geq 2$ . It has  $a(T) = n(T) - 1$ ,  $\alpha(T) = n(T) - 1$ , and  $\mu(T) = 1$ . Thus, we can observe that  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$ . □

**Lemma 2.7.** *There exists an infinite family of graphs that satisfy the equality  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$  with  $\mu(T) = 3$ .*

*Proof.* For graph with  $n$  vertices ( $n \geq 6$ ) having the structure shown in Fig. 3, the degree sequence can be represented as follows:

$$\underbrace{1, \dots, 1}_{n-3}, \underbrace{2, 2}_2, \underbrace{n-3}_1.$$

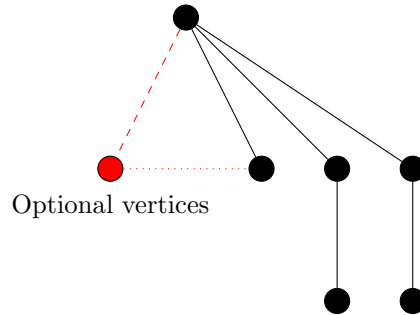


Figure 3: An example of a graph with  $\mu(T) = 3$  where the tight bound is achieved.

For such graphs, it has  $m(T) = n(T) - 1$ ,  $a(T) = n(T) - 2$ ,  $\alpha(T) = n(T) - 3$ , and  $\mu(T) = 3$ . This configuration satisfies the tight bound  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$ . □

**Lemma 2.8.** *There exists an infinite family of graphs that satisfy the equality  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$  with  $\mu(T) = 5$ .*

*Proof.* For any graph with  $n$  vertices ( $n \geq 10$ ) having the structure shown in Fig. 4, the degree sequence can be represented as follows:

$$\underbrace{1, \dots, 1}_{n-5}, \underbrace{2, \dots, 2}_4, \underbrace{n-5}_1.$$

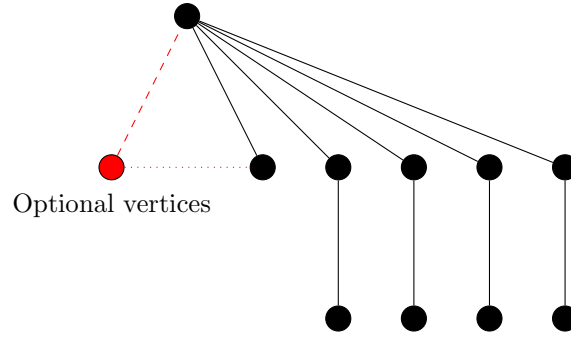


Figure 4: An example of a graph with  $\mu(T) = 5$  where the tight bound is achieved.

For such graphs, it has  $m(T) = n(T) - 1$ ,  $a(T) = n(T) - 3$ ,  $\alpha(T) = n(T) - 5$ , and  $\mu(T) = 5$ . Thus, this configuration satisfies the tight bound  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$ .  $\square$

**Theorem 2.9.** *There exists an infinite family of graphs that satisfy the equality  $a(T) - \alpha(T) = \frac{\mu(T)}{2} - \frac{1}{2}$  with  $\mu(T) = 2q + 1$ , where  $q$  is a positive integer.*

*Proof.* We can deduce the following from the previous three lemmas:

- The number of vertices is  $n(T) = 4q + 2$ ,
- The independence number is  $\alpha(T) = 2q + 1$ ,
- There is a root of degree  $2q + 1$ ,
- There are  $2q$  vertices of degree 2,
- There are  $2q + 1$  vertices with degree 1.

$T$  is a tree, so  $m(T) = 4q + 1$ . Now, let us consider the sum of the degrees of the vertices until we reach  $m(T)$ . There are  $2q + 1$  vertices that are leaves, and we select  $q$  vertices from the  $2q$  vertices with degree 2. Therefore, the sum becomes:

$$(2q + 1) \cdot 1 + q \cdot 2 = 4q + 1$$

and  $a(T) = 3q + 1$ . Substituting these values into the equation, we get,

$$\underbrace{3q + 1}_{a(T)} - \underbrace{2q + 1}_{\alpha(T)} = \underbrace{\frac{2q + 1}{2}}_{\frac{\mu(T)}{2}} - \frac{1}{2}.$$

□

**Observation 2.10.**  $\mu(T)$  can only be an odd number in this case. This is because the difference on the left side of the equality,  $a(T) - \alpha(T)$ , is an integer, and the right side,  $\frac{\mu(T)}{2} - \frac{1}{2}$ , must also be an integer. Since  $\frac{\mu(T)}{2} - \frac{1}{2}$  is not an integer when  $\mu(T)$  is even, the only possibility is for  $\mu(T)$  to be an odd number.

**Lemma 2.11.** There exist bipartite graphs that do not satisfy the inequality  $a(G) - \alpha(G) \leq \frac{\mu(G)}{2} - \frac{1}{2}$ .

*Proof.* Let us take a bipartite graph  $G$  with  $V(G) = A \cup B \cup C$ , where  $|A| = 16, |B| = 8, |C| = 8$ .  $A$  (red vertices in Figure 5) and  $B$  (blue vertices in Figure 5) are independent sets, and  $C$  (black vertices in Figure 5) is a complete bipartite graph  $K_{4,4}$ . Half of the vertices from  $A$  are connected to  $B$  by a matching, while another half of  $A$  are connected to  $C$  by a matching. Additionally, each vertex from  $B$  is connected to one vertex from  $C$ .

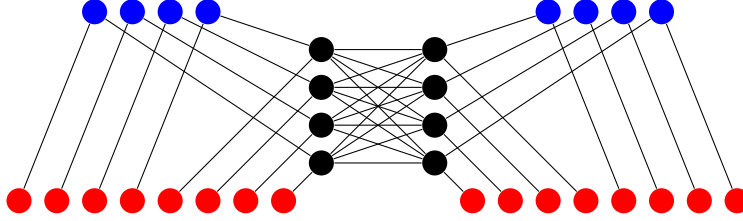


Figure 5: A bipartite graph not satisfying the inequality for trees.

In this given example, there is a bipartite graph with  $n = 32$  vertices and  $a(G) = 25, \alpha(G) = 16, \mu(G) = 16$  and  $25 - 16 > \frac{16-1}{2}$ . □

Given that every tree is a König–Egerváry graph, Lemma 1.4 can be applied that for trees  $\mu(T) \leq \alpha(T)$ . Now, we can express the tight bound for trees solely in terms of the annihilation number  $a(T)$  and the independence number  $\alpha(T)$  as follows.

**Corollary 2.12.** If  $T$  is a tree, then  $a(T) \leq \frac{3}{2}\alpha(T) - \frac{1}{2}$ .

*Proof.*

$$\begin{aligned} a(T) - \alpha(T) &\leq \frac{\mu(T)}{2} - \frac{1}{2} \leq \frac{\alpha(T)}{2} - \frac{1}{2} \\ a(T) &\leq \frac{3}{2}\alpha(T) - \frac{1}{2}. \end{aligned}$$

□



### 3 The tight upper bound for bipartite graphs

To establish the tight upper bound for bipartite graphs, we make use of the following well-known lemma.

**Lemma 3.1.** *The maximum number of edges in a bipartite graph  $G$  is  $\frac{n(G)^2}{4}$ .*

**Theorem 3.2.** *For every bipartite graph  $G$ ,  $a(G) - \alpha(G) \leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$ .*

*Proof.* Consider a bipartite graph  $G$  with an annihilation decomposition  $\langle A, B \rangle$ .

$$\text{Deg}(A) = 2m(G[A]) + k\langle A, B \rangle \leq m(G) \leq 2m(G[B]) + k\langle A, B \rangle = \text{Deg}(B) \quad (\text{Lemma 2.2})$$

$$2(a(G) - \alpha(G)) + k\langle A, B \rangle \leq 2m(G[A]) + k\langle A, B \rangle \quad (\text{Lemma 2.3})$$

$$2m(G[B]) + k\langle A, B \rangle \leq 2 \left( \frac{(n(G) - a(G))^2}{4} \right) + k\langle A, B \rangle \quad (\text{Lemma 3.1})$$

Combining the above inequalities, we obtain

$$\begin{aligned} 2(a(G) - \alpha(G)) + k\langle A, B \rangle &\leq 2m(G[A]) + k\langle A, B \rangle \leq m(G) \leq \\ &\leq 2m(G[B]) + k\langle A, B \rangle \leq 2 \left( \frac{(n(G) - a(G))^2}{4} \right) + k\langle A, B \rangle. \end{aligned}$$

Hence,

$$2(a(G) - \alpha(G)) + k\langle A, B \rangle \leq 2 \left( \frac{(n(G) - a(G))^2}{4} \right) + k\langle A, B \rangle$$

$$a(G) - \alpha(G) \leq \frac{(n(G) - a(G))^2}{4}$$

$$4(a(G) - \alpha(G)) \leq (\alpha(G) - a(G) + \mu(G))^2$$

$$4(a(G) - \alpha(G)) \leq (\alpha(G) - a(G))^2 - 2\mu(G)(a(G) - \alpha(G)) + \mu(G)^2$$

$$(a(G) - \alpha(G))^2 - (2\mu(G) + 4)(a(G) - \alpha(G)) + \mu(G)^2 \geq 0$$

The zeros of the equation are:

$$(a(G) - \alpha(G))_{1,2} = 2 + \mu(G) \pm 2\sqrt{1 + \mu(G)}.$$

In the last equality, we look for values that are bigger or equal to zero, so we have only two cases,

$$\textbf{Case 1: } a(G) - \alpha(G) \geq 2 + \mu(G) + 2\sqrt{1 + \mu(G)}$$

By Lemma 1.3, For König–Egerváry graphs  $a(G) - \alpha(G) \leq \mu(G)$ . Hence, this case is impossible.

$$\textbf{Case 2: } a(G) - \alpha(G) \leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$$

By Lemma 1.3,  $a(G) - \alpha(G) \leq \mu(G)$ , and easy to see that  $2 - 2\sqrt{1 + \mu(G)}$  is a negative number. So this case holds for every  $\mu(G) > 0$ .  $\square$

**Theorem 3.3.** *The inequality  $a(G) - \alpha(G) \leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$  is the tight bound for bipartite graphs.*

*Proof.* As an illustration, Figure 6 shows the case of a bipartite graph with  $n(G) = 16$ , where  $a(G) = 12$ ,  $\alpha(G) = 8$ , and  $\mu(G) = 8$ . Notably, this graph satisfies the equality corresponding to the tight bound:  $12 - 8 = 2 + 8 - 2\sqrt{1 + 8}$  and not satisfies the tight bound for trees:  $12 - 8 > \frac{8}{2} - \frac{1}{2}$ .

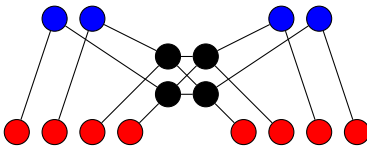


Figure 6: A bipartite graph that satisfies the equality.

□

**Theorem 3.4.** *There exists an infinite family of graphs where the difference between the annihilation number  $a(G)$  and the independence number  $\alpha(G)$  is equal to  $2 + \mu(G) - 2\sqrt{1 + \mu(G)}$ .*

*Proof.* Let  $p \geq 3$  be an integer. By observing Figure 6, it becomes apparent that the vertex set can be divided into three distinct sets:

- A maximum independent set, denoted as  $A$ , which consists of  $p^2 - 1$  vertices. In this set, we have  $(p - 1)^2$  vertices connected to  $B$  in a one-to-one manner and  $2p - 2$  vertices connected to  $C$  in a one-to-one manner. All the vertices in set  $A$  have a degree of 1, meaning that each vertex is connected to exactly one other vertex in the graph.
- Similarly, independent set denoted as  $B$ , which contains  $(p - 1)^2$  vertices. In this set, all the vertices in  $B$  are connected to vertices in  $A$  in a one-to-one manner. Additionally,  $2p - 2$  vertices from set  $B$  are also connected to all the vertices in  $C$  in a one-to-one manner. As a result, there are  $p^2 - 4p + 3$  vertices in  $B$  with a degree of 1. Moreover, there are  $2p - 2$  vertices in  $B$  with a degree of 2, as they are connected to both vertices in  $A$  and vertices in  $C$ .
- Set  $C$ , which is a complete bipartite graph  $K_{p-1, p-1}$  consisting of  $2p - 2$  vertices. In this set, every vertex in  $C$  is connected to one vertex in  $A$  and one vertex in  $B$ , forming one-to-one connections. Consequently, all vertices in set  $C$  have a degree equal to  $p + 1$ .

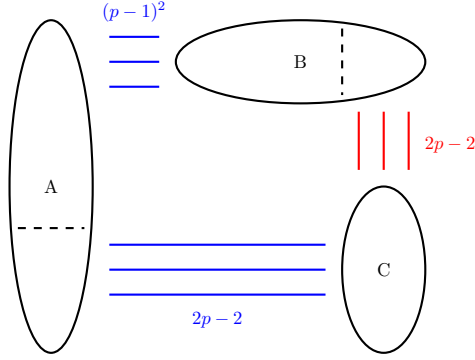


Figure 7: The partition of bipartite graphs where the tight bound is achieved.

The degree sequence is  $\underbrace{1, \dots, 1}_{p^2 - 1}, \underbrace{1, \dots, 1}_{p^2 - 4p + 3}, \underbrace{2, \dots, 2}_{2p - 2}, \underbrace{p + 1, \dots, p + 1}_{2p - 2}$ .

To ensure that we avoid negative values for  $p^2 - 4p + 3$ , it is suggested that  $p \geq 3$ .

In this partition, we observe that  $\alpha(G) = p^2 - 1 = |A|$  and  $\mu(G) = p^2 - 1 = |A|$ . Now,  $m(G)$  can be calculated as follows:

$$\begin{aligned} m(G) &= \frac{1 \cdot (p^2 - 1) + 1 \cdot (p^2 - 4p + 3) + 2 \cdot (2p - 2) + (2p - 2) \cdot (p + 1)}{2} \\ &= 2p^2 - 2. \end{aligned}$$

Let us calculate  $a(G)$ . We consider all the vertices in sets  $A$  and  $B$ . These vertices have degrees of either 1 or 2.

$$\underbrace{1, \dots, 1, 1, \dots, 1, 2, \dots, 2}_{p^2 - 1}, p + 1, \dots, p + 1.$$

Now,

$$1 \cdot (p^2 - 1) + 1 \cdot (p^2 - 4p + 3) + 2 \cdot (2p - 2) = 2p^2 - 2 = m(G).$$

Hence, the maximum value that  $a(G)$  can be is given by,

$$a(G) = (p^2 - 1) + (p^2 - 4p + 3) + (2p - 2) = 2p^2 - 2p.$$

And, the difference between  $a(G)$  and  $\alpha(G)$  is,

$$a(G) - \alpha(G) = 2p^2 - 2p - (p^2 - 1) = p^2 - 2p + 1 = (p - 1)^2.$$

Another way to calculate this difference is by setting  $\mu(G)$  to be  $p^2 - 1$  on the right side of the bipartite inequality bound. We have

$$2 + \mu(G) - 2\sqrt{1 + \mu(G)} = 2 + (p^2 - 1) - 2\sqrt{1 + (p^2 - 1)} = p^2 - 2p + 1 = (p - 1)^2.$$

As we can see, we obtained the same expression for the difference between  $a(G)$  and  $\alpha(G)$ . □

**Observation 3.5.** *If  $p = 3$ , then the graph  $G$  is connected. On the other hand, if  $p \geq 4$ , then the graph  $G$  is disconnected.*

**Lemma 3.6.** *There exist a König–Egerváry graph that do not satisfy the inequality  $a(G) - \alpha(G) \leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$ .*

*Proof.* In the graph shown in Figure 8, there are 12 vertices. The blue vertices represent the independent set with an independence number of 6. The red vertices represent the matching with a matching number also equal to 6. Therefore,  $G$  is a König–Egerváry graph. The degree sequence of this graph is:

$$1, 2, 2, 2, 3, 3, 3, 3, 3, 7, 7, 8$$

and the annihilation number is 9. In this example, we have  $a(G) - \alpha(G) = 9 - 6 = 3$ , which is greater than the value obtained from the inequality  $2 + \mu(G) - 2\sqrt{1 + \mu(G)} = 2 + 6 - 2\sqrt{1 + 6} = 2.7$ .

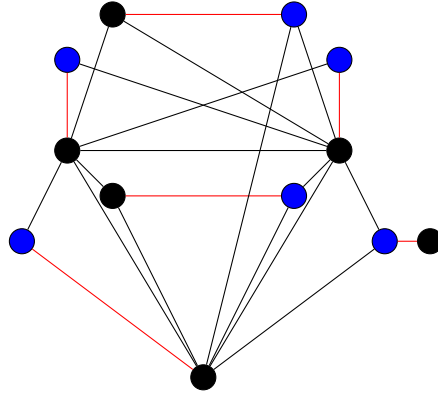


Figure 8: A König–Egerváry graph not satisfying the bipartite tight bound. □

By utilizing Lemma 1.4, we can rephrase the bipartite inequality to solely involve  $a$  and  $\alpha$  as follows.

**Corollary 3.7.** *If  $G$  is a bipartite graph, then  $a(G) \leq 2 + 2\alpha(G) - 2\sqrt{1 + \alpha(G)}$ .*

*Proof.*

$$\begin{aligned} a(G) - \alpha(G) &\leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)} \leq 2 + \alpha(G) - 2\sqrt{1 + \alpha(G)} \\ a(G) &\leq 2 + 2\alpha(G) - 2\sqrt{1 + \alpha(G)}. \end{aligned}$$

□

## 4 The tight upper bound for König–Egerváry graphs

To establish the tight upper bound for König–Egerváry graphs, we present the following lemmas.

**Lemma 4.1.** *If a graph  $G$  is connected, the inequality  $a(G) \leq n(G) - 1$  holds only for star graphs.*

*Proof.* Consider a graph  $G$  with an annihilation number  $a(G)$  equal to  $n(G) - 1$ . The sum of the degrees of the first  $n(G) - 1$  vertices is equal to the degree of the last vertex in the degree sequence. However, the maximum degree of the last vertex can only be  $n(G) - 1$ , which is the case for star graphs.  $\square$

**Corollary 4.2.** *For every connected graph that is not a star graph, the annihilation number  $a(G)$  is bounded by  $n(G) - 2$ .*

**Theorem 4.3.** *For every König–Egerváry graph, the inequality  $a(G) - \alpha(G) \leq \mu(G) - 2$  represents the tight bound.*

*Proof.* Consider a König–Egerváry graph  $G$  that is not a bipartite graph, with the following structure: Start with a cycle  $C_3$  containing vertices  $v_1$ ,  $v_2$ , and  $v_3$ . Vertex  $v_1$  is connected to an independent set of size  $k$ , vertex  $v_2$  is connected to another independent set of size  $k$ , and vertex  $v_3$  is connected to a single additional vertex. This graph has a total of  $2k + 4$  vertices,  $\alpha(G) = 2k + 1$ , and  $\mu(G) = 3$ .

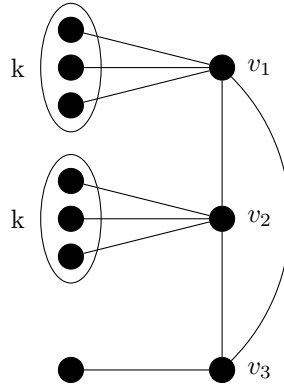


Figure 9: Family of infinity graphs that satisfy the tight bound with  $\mu = 3$

In this structure, the degree sequence is,

$$\underbrace{1, \dots, 1}_{2k+1}, \underbrace{3}_1, \underbrace{k+2}_1, \underbrace{k+2}_1$$

The number of edges in each graph is equal to  $2k+4$ . The annihilation number  $a(G)$  includes all the vertices with a degree of 1 or 3, so  $a(G) = 2k+2$ . Hence, we have  $a(G) = n(G) - 2$ , which can be expressed differently as  $a(G) - \alpha(G) = \mu(G) - 2$ .

□

For example, Figure 10 illustrates the configuration from Figure 9 with  $k = 2$ .

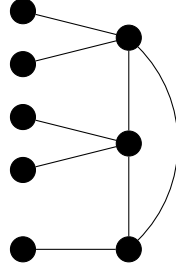


Figure 10: Example of a König–Egerváry graph with  $n = 8$  and  $a = 6$ .

Based on Theorem 4.3 and Lemma 1.4, we can rephrase the König-Egerváry inequality to involve only  $a(G)$  and  $\alpha(G)$  as follows:

**Corollary 4.4.** *For König–Egerváry graphs that are not bipartite, we have  $a(G) \leq 2\alpha(G) - 2$ .*

*Proof.*

$$\begin{aligned} a(G) - \alpha(G) &\leq \mu(G) - 2 \leq \alpha(G) - 2, \\ a(G) &\leq 2\alpha(G) - 2. \end{aligned}$$

□

## 5 Conclusions

This paper investigates three tight bounds on the difference between the annihilation number  $a(G)$  and the independence number  $\alpha(G)$ . These bounds are as follows:

- For every tree  $T$ , we have  $a(T) - \alpha(T) \leq \frac{\mu(T)}{2} - \frac{1}{2}$ .
- For every bipartite graph  $G$ , we have  $a(G) - \alpha(G) \leq 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$ .
- For every König–Egerváry graph  $G$ , we have  $a(G) - \alpha(G) \leq \mu(G) - 2$ .

Those three inequalities can be rewritten by Corollaries 2.12, 3.7 and 4.4 by the following.

- For every tree  $T$ ,  $a(T) \leq \frac{3}{2}\alpha(T) - \frac{1}{2}$ .

- For every bipartite graph  $G$ ,  $a(G) \leq 2 + 2\alpha(G) - 2\sqrt{1 + \alpha(G)}$ .
- For every König–Egerváry graph  $G$ ,  $a(G) \leq 2\alpha(G) - 2$ .

**Problem 5.1.** Find the tight bound on the difference between  $a(G)$  and  $\alpha(G)$  for forests.

**Conjecture 5.2.** There exists an infinite family of connected bipartite graphs that illustrate the tight bound in Theorem 3.2.

**Conjecture 5.3.** The minimum number of vertices for a bipartite graph satisfying the equality  $a(G) - \alpha(G) = 2 + \mu(G) - 2\sqrt{1 + \mu(G)}$  is 16.

**Problem 5.4.** Mantel’s theorem states that for triangle-free graphs, the number of edges  $m$  is at most  $\frac{n(G)^2}{4}$ , which is the same bound as for bipartite graphs. Find the tight upper bound for the difference between  $a(G)$  and  $\alpha(G)$  for triangle-free graphs.

**Problem 5.5.** Find the tight upper bound for the difference between  $a(G)$  and  $\alpha(G)$  for general graphs.

## References

- [1] J. Amjadi, *An upper bound on the double domination number of trees*, Kragujevac J. Math. 39 (2015), 133–139, <https://doi.org/10.5937/KgJMath1502133A>
- [2] H. Aram, R. Khoelilar, S. M. Sheikholeslami and L. Volkmann, *Relating the annihilation number and the Roman domination number*, Acta Math. Univ. Comenian. (N.S.) 87 (2018), 1–13
- [3] E. Boros, M. C. Golumbic and V. E. Levit, *On the number of vertices belonging to all maximum stable sets of a graph*, volume 124, pp. 17–25 (2002), [https://doi.org/10.1016/S0166-218X\(01\)00327-4](https://doi.org/10.1016/S0166-218X(01)00327-4), workshop on Discrete Optimization (Piscataway, NJ, 1999).
- [4] C. Bujtás and M. Jakovac, *Relating the total domination number and the annihilation number of cactus graphs and block graphs*, Ars Math. Contemp. 16 (2019), 183–202, <https://doi.org/10.26493/1855-3974.1378.11d>
- [5] N. Dehgardi, S. Norouzian and S. M. Sheikholeslami, *Bounding the domination number of a tree in terms of its annihilation number*, Trans. Comb. 2 (2013), 9–16, <https://doi.org/10.22108/TOC.2013.2652>
- [6] N. Dehgardi, S. M. Sheikholeslami and A. Khodkar, *Bounding the rainbow domination number of a tree in terms of its annihilation number*, Trans. Comb. 2 (2013), 21–32, <https://doi.org/10.22108/TOC.2013.3051>.

- [7] N. Dehgard, S. M. Sheikholeslami and A. Khodkar, *Bounding the paired-domination number of a tree in terms of its annihilation number*, *Filomat* 28 (2014), 523–529, <https://doi.org/10.2298/FIL1403523D>
- [8] R. W. Deming, *Independence numbers of graphs—an extension of the Koenig-Egervary theorem*, *Discrete Math.* 27 (1979), 23–33, [https://doi.org/10.1016/0012-365X\(79\)90066-9](https://doi.org/10.1016/0012-365X(79)90066-9).
- [9] W. J. Desormeaux, T. W. Haynes and M. A. Henning, *Relating the annihilation number and the total domination number of a tree*, *Discrete Appl. Math.* 161 (2013), 349–354, <https://doi.org/10.1016/j.dam.2012.09.006>
- [10] E. Egervary, *On combinatorial properties of matrices*, *Matematikai Lapok*, 38 (1931).
- [11] M. Gentner, M. A. Henning and D. Rautenbach, *Smallest domination number and largest independence number of graphs and forests with given degree sequence*, *J. Graph Theory* 88 (2018), 131–145, <https://doi.org/10.1002/jgt.22189>
- [12] X. Hua, K. Xu and H. Hua, *Relating the annihilation number and the total domination number for some graphs*, *Discrete Applied Mathematics* 332 (2023) 41–46, <https://doi.org/10.1016/j.dam.2023.01.018>
- [13] M. Jakovac, *Relating the annihilation number and the 2-domination number of block graphs*, *Discrete Appl. Math.* 260 (2019), 178–187, <https://doi.org/10.1016/j.dam.2019.01.020>
- [14] D. A. Jaumea and G. Molina, *Maximum and minimum nullity of a tree degree sequence* (2018), <https://arxiv.org/abs/1806.02399>
- [15] D. Konig, *Graphen und matrizen*, *Matematikai Lapok*, 38 (1931), 116–119.
- [16] C. E. Larson and R. Pepper, *Graphs with equal independence and annihilation numbers*, *Electron. J. Combin.* 18 (2011), Paper 180, 9, <https://doi.org/10.37236/667>.
- [17] V. E. Levit, E. Mandrescu, *On an annihilation number conjecture*, *ARS Mathenatica Contemporanea*, (2020), ISSN 1855-3966 (printed edn.), ISSN 1855-3974 (electronic edn.) <https://doi.org/10.26493/1855-3974.1950.8bd> (Also available at <http://amc-journal.eu>)
- [18] V. E. Levit, E. Mandrescu, *Some More Updates on an Annihilation Number Conjecture: Pros and Cons*, *Graphs and Combinatorics* (2022) 38:141 <https://doi.org/10.1007/s00373-022-02534-7>
- [19] W. Ning, M. Lu and K. Wang, *Bounding the locating-total domination number of a tree in terms of its annihilation number*, *Discuss. Math. Graph Theory* 39 (2019), 31–40, [doi.org/10.7151/dmgt.2063](https://doi.org/10.7151/dmgt.2063)



- [20] R. Pepper, *On the annihilation number of a graph*, Recent Advances in Electrical Engineering, Proceedings of the 15th American Conference on Applied Mathematics (2009), pp. 217–220.
- [21] R. D. Pepper, *Binding independence*, ProQuest LLC, Ann Arbor, MI (2004), thesis (Ph.D.) University of Houston, [http://gateway.proquest.com/openurl?url\\_ver=Z39.88-2004&rft\\_val\\_fmt=info:ofi/fmt:kev:mt](http://gateway.proquest.com/openurl?url_ver=Z39.88-2004&rft_val_fmt=info:ofi/fmt:kev:mt)
- [22] F. Sterboul, *A characterization of the graphs in which the transversal number equals the matching number*, J. Combin. Theory Ser. B, 7 (1979), 228–229, [https://doi.org/10.1016/0095-8956\(79\)90085-6](https://doi.org/10.1016/0095-8956(79)90085-6).