
EFFICIENT NEARLY-FAIR DIVISION WITH CAPACITY CONSTRAINTS

DEPARTMENT OF COMPUTER SCIENCE, ARIEL UNIVERSITY, ARIEL 40700, ISRAEL

Hila Shoshan
hilashoshan0605@gmail.com

Erel Segal-Halevi
erelsgl@gmail.com

Noam Hazon
noam.hazon@g.ariel.ac.il

ABSTRACT

We consider the problem of fairly and efficiently matching agents to indivisible items, under capacity constraints. In this setting, we are given a set of categorized items. Each category has a capacity constraint (the same for all agents), that is an upper bound on the number of items an agent can get from each category. Our main result is a polynomial-time algorithm that solves the problem for two agents with additive utilities over the items. When each category contains items that are all goods (positively evaluated) or all chores (negatively evaluated) for each of the agents, our algorithm finds a feasible allocation of the items, which is both Pareto-optimal and envy-free up to one item. In the general case, when each item can be a good or a chore arbitrarily, our algorithm finds an allocation that is Pareto-optimal and envy-free up to one good and one chore.

1 Introduction

The problem of how to fairly divide a set of indivisible items among agents is of central importance, and it has been vastly investigated by mathematicians, economists, political scientists and computer scientists. Most of the early work on fair division focused on how to fairly divide goods, i.e., items with non-negative utility. In recent years, several works have considered the division of chores, i.e., items with negative utility, and a few works also considered the division of a mixture of goods and chores (for example, Aziz et al. (2022) and Bérczi et al. (2020)).

Indeed, in many situations there is a mixture of goods and chores. Moreover, there are situations in which several items may be considered as goods for one agent and as chores for another agent. For example, consider a final project that has to be completed by a team of students. It consists of several tasks that should be matched to the students: 5 programming tasks, 6 user-interface tasks, and 3 algorithmic tasks. One student may evaluate the programming tasks as items with negative utilities and the algorithmic tasks as items with positive utilities, while another student may evaluate them the other way around.

As common in many other fields, real-life situations give rise to important insights that should be considered by fair-division algorithms. In particular, real-life situations typically add some constraints that should be taken into account (see Suksompong (2021) for an overview of different types of constraints). Arguably, a very common constraint is that the items belong to different

categories, and each category has an associated capacity constraint, which defines the maximum number of tasks that it is allowed to assign to each agent. For example, consider again the final project example, and suppose that the team consists of 2 students. The mentor of the project would like that all the members of the team will gain knowledge of the different development tasks, and thus they must be involved in all of the aspects of the project. Therefore, the mentor of the project should define 3 categories of the project’s tasks: programming, user-interface, and algorithms. Then, the mentor should set a capacity of 3 for the programming and user-interface categories, and a capacity of 2 for the algorithms’ category, for ensuring that both students will be involved in about the same number of tasks from each category. Clearly, the capacity constraints should be large enough so that all of the items in a given category could be assigned to the agents.

The constraint of categories with capacities have been previously investigated when the items are goods (for example, by Biswas and Barman (2018) and Dror et al. (2021)), but, for the best of our knowledge, it has not been considered when the items are chores, or when there is a mixture of goods and chores, which is the focus of this paper.

We assume that the items are divided into categories. Each category has a capacity constraint (the same for all agents), that is an upper bound on the number of items an agent can get from each category. We focus on the setting in which agents have additive utilities over the items, and there are two agents. Indeed, one agent may evaluate an item in a given category as a good, while the other agent may evaluate it as a chore. Note that without capacity constraints, we can simply give all such items to the agent who evaluates them as goods, as done by Aziz et al. (2022), but with capacity constraints this may not be possible.

As an efficiency criterion, we use Pareto optimality (PO), which means that no other allocation is at least as good for all agents and strictly better for some agent. As fairness criteria, we use two relaxations of envy-freeness. The stronger one is *envy-freeness up to one item (EF1)*, which was introduced by Budish (2011), and adapted by Aziz et al. (2022) for the setting of mixture of goods and chores. Intuitively, an allocation is EF1 if for each pair of agents i, j , after removing the most difficult chore (for i) from i ’s bundle, *or* the most valuable good (for i) from j ’s bundle, i would not be jealous of j .

With capacity constraints, an EF1 allocation may not exist. For example, consider a scenario with one category with two items, o_1 and o_2 , and capacity constraint of 1. o_1 is a good for both agents (e.g., $u_1(o_1) = u_2(o_1) = 1$), and o_2 is a chore for both agents (e.g., $u_1(o_2) = u_2(o_2) = -1$). Clearly, in every feasible allocation, one agent must receive the good and the other agent must receive the chore (due to the capacity constraint), and thus the allocation is not EF1. Therefore, we introduce a natural relaxation of it, called *envy-freeness up to one good and one chore (EF[1,1])*, which means that for each pair of agents i, j , after removing both the most difficult chore (for i) from i ’s bundle, *and* the most valuable good (for i) from j ’s bundle, i would not be jealous of j .

We provide a polynomial time algorithm that, for two agents, returns an allocation that is both PO and EF[1,1]. In the special case in which, for each agent and category, either all items are goods or all items are chores, the allocation is also EF1.

Our algorithm is based on the following ideas. The division problem can be considered as a matching problem on a bipartite graph, in which one side represents the agents and the other side represents the items. We add dummy items and clones of agents such that the capacity constraints are guaranteed. We assign a positive weight to each agent. We assign, to each edge between an agent and an item, a weight which is the product of the agent weight and the valuation of the

agent to the item. A maximum-weight matching in this graph represents a feasible allocation that maximizes a weighted sum of utilities. Every allocation that maximizes a weighted sum of utilities, with positive agent weights, is Pareto-optimal.¹ Our algorithm first computes a maximum-weight matching that is also envy-free for one of the agents. It then tries to make it EF1 (or EF[1,1]), while maintaining it a maximum-weight matching, by identifying pairs of items that can be exchanged between the agents, based on a ratio that captures how much one agent prefers an item relative to the other agent’s preferences. Every exchange of items is equivalent to increasing the jealous agent’s weight and decreasing the other agent’s weight. Therefore, our technique resembles the moving-knife technique that is used when dividing a divisible item (i.e., a cake), but our algorithm requires only a discrete number of steps, which is polynomial in the number of items.

2 Related Work

Fair division problems vary according to the nature of the objects being divided, the preferences of the agents, and the fairness criteria. Many algorithms have been developed to solve fair division problems, for details see the surveys of such algorithms (Brams and Taylor, 1996; Moulin, 2004; Brams, 2007; Bouveret et al., 2016).

2.1 Mixtures of Goods and Chores

Bérczi et al. (2020) present a polynomial-time algorithm for finding an EF1 allocation for two agents with arbitrary utility functions (positive or negative). Chen and Liu (2020) proved that the leximin solution is EFX for combinations of goods and chores for agents with identical valuations. Gafni et al. (2021) present a generalization of both goods and chores, by considering items that may have several copies. All these works do not consider efficiency.

Efficiency in a setting with goods and chores is studied by Aziz et al. (2022). They use the round-robin technique for finding an EF1 and PO division of combinations of goods and chores between two agents. Similarly, Aziz et al. (2020) find an allocation that is PROP1 and PO for goods and chores. Aleksandrov and Walsh (2019) prove that, with tertiary utilities, EFX and PO allocations always exist for mixed items. However, all of these works do not handle capacity constraints.

2.2 Constraints

When all agents have weakly additive utilities, the round-robin protocol finds a complete EF1 division in which all agents receive approximately the same number of items (Caragiannis et al., 2019). This technique, together with the envy-graph, has been used for finding a fair division of goods under capacity constraints (Biswas and Barman, 2018). This work has been extended to heterogeneous capacity constraints (Dror et al., 2021), and to maximin-share fairness (Hummel and Hetland, 2021).

Fair allocation of goods of different categories has been studied by Mackin and Xia (2016) and Sikdar et al. (2017). Each category contains n goods, and each agent must receive exactly one item of each category. Sikdar et al. (2019) consider an exchange market in which each agent holds multiple items of each category and should receive a bundle with exactly the same number of items of each category. All of these works consider only goods.

¹In fact, maximizing a weighted sum of utilities is stronger than Pareto-optimality. When allocating goods without capacity constraints, maximizing a weighted sum of utilities is equivalent to a stronger efficiency notion called *fractional Pareto-optimality* (Negishi, 1960; Varian, 1976; Barman et al., 2018a).

Nyman et al. (2020) study a similar setting (they call the categories “houses” and the items “rooms”), but with monetary transfers (which they call “rent”).

Several other constraints have been considered. For example, Bilò et al. (2022) study the fair division of goods such that each bundle needs to be connected on an underlying graph. Igarashi and Peters (2019) study PO allocation of goods with connectivity constraints. An overview of the different types of constraints that have been considered can be found in (Suksompong, 2021).

2.3 Efficiency and Fairness

There are several techniques for finding a division of goods that is EF1 and PO. For example, the Maximum Nash Welfare algorithm selects a complete allocation that maximizes the product of utilities. It assumes that the agents’ utilities are additive, and the resulting allocation is both EF1 and PO. (Caragiannis et al., 2019; Wu et al., 2020). Barman et al. (2018b) present a price-based mechanism that finds an EF1 and PO allocation of goods in pseudo-polynomial time.

Finally, in the context of fair cake-cutting (fair division of a continuous resource), Weller (1985) proved the existence of an EF and PO allocation by considering the set of all allocations that maximize a weighted sum of utilities. We adapted this technique for the setting with indivisible items and capacity constraints.

2.4 Alternative techniques

Our setting combines a mixture of goods and chores, capacity constraints, and a guarantee of both fairness and efficiency. These three issues were studied in separation, but not all simultaneously.

Although previous works have developed useful techniques, they do not work for our setting. For example, using the top-trading graph presented by Bhaskar et al. (2021) for dividing chores does not work when there are capacity constraints. The reason is that if we allocate an item to the “sink” agent (i.e., an agent that does not envy any agent) on the top-trading graph, we may exceed the capacity constraints. As another example, consider the maximum-weighted matching algorithm of Brustle et al. (2020). It is not hard to modify the algorithm to work with chores, but adding capacity constraints on each category might not maintain the EF1 property between the categories. See Appendix A for more details.

Therefore, in this paper we develop a new technique for finding PO and EF1 (or EF[1,1]) allocation of the set of items, that also maintains capacity constraints.

3 Notations

An instance of our problem is a tuple $I = (N, M, C, S, U)$ where:

- $N = [n]$ is a set of agents.
- $M = (o_1, \dots, o_m)$ is a set of items.
- $C = (C_1, C_2, \dots, C_k)$ is a set of k categories. The categories are pairwise-disjoint and $M = \bigcup_j C_j$.
- $S = \{s_1, s_2, \dots, s_k\}$ is a list of size k , containing the capacity constraint of each category. We assume that $\forall j \in [k]: \frac{|C_j|}{n} \leq s_j \leq |C_j|, s_j \in \mathbb{N}$. The lower bound is needed to ensure we

can divide all the items, and not "throw" anything away, and the upper bound is a trivial bound specified to ensure good running time.

- U is an n -tuple of utility functions $u_i : M \rightarrow \mathbb{R}$. We assume additive utilities, that is, $u_i(X) := \sum_{o \in X} u_i(o)$ for $X \subseteq M$.

In a general *mixed* instance, each utility can be any real number (positive, negative or zero). A *same-sign instance* is an instance in which, for each agent $i \in N$ and category $j \in [k]$, C_j contains only goods for i or only chores for i . That is, either $u_i(o) \geq 0$ for all $o \in C_j$, or $u_i(o) \leq 0$ for all $o \in C_j$. Note that, even in a same-sign instance, it is possible that each agent evaluates different categories as goods or chores, and that two different agent evaluate the same item differently.

An *allocation* is a vector $A := (A_1, A_2, \dots, A_n)$, with $\forall i, j \in [n], i \neq j : A_i \cap A_j = \emptyset$ and $\bigcup_{i \in [n]} A_i = M$. A_i is called "agent i 's bundle". An allocation A is called *feasible* if for all $i \in [n]$, the bundle A_i contains at most s_c items of each category C_c .

Definition 3.1 (Due to Aziz et al. (2022)). An allocation A is called *Envy Free up to one item (EF1)* if for all $i, j \in N$, at least one of the following holds:

- There exists a set $T \subseteq A_i$ with $|T| \leq 1$, s.t. $u_i(A_i \setminus T) \geq u_i(A_j)$.
- There exists a set $G \subseteq A_j$ with $|G| \leq 1$, s.t. $u_i(A_i) \geq u_i(A_j \setminus G)$.

We also define a slightly weaker fairness notion, that we need for handling general mixed instances, in which an EF1 allocation is not guaranteed to exist.

Definition 3.2. An allocation A is called *Envy Free up to one good and one chore (EF[1,1])* if for all $i, j \in N$, there exists a set $T \subseteq A_i$ with $|T| \leq 1$, and a set $G \subseteq A_j$ with $|G| \leq 1$, such that $u_i(A_i \setminus T) \geq u_i(A_j \setminus G)$.

Finally, we recall the efficiency criterion:

Definition 3.3. Given an allocation A , another allocation A' is a *Pareto-improvement* of A if $u_i(A'_i) \geq u_i(A_i)$ for all $i \in N$, and $u_j(A'_j) > u_j(A_j)$ for some $j \in N$.

A feasible allocation A is *Pareto-Optimal (PO)* if no feasible allocation is a Pareto-improvement of A .

4 Finding an Efficient and Nearly-Fair Division

In this section, we present some general notions that can be used for any number of agents. Then, we present our algorithm that finds in polynomial time a PO division with two agents. In any mixed instance, this division is also EF[1,1]; in a same-sign instance, it is also EF1.

4.1 Preprocessing

We preprocess the instance such that, in any feasible allocation, both bundles have the same cardinality. To achieve this, we add to each category C_c with capacity constraint s_c , some $ns_c - |C_c|$ dummy items with a value of 0 to all agents. In the new instance, each bundle must contain exactly s_c items from each category C_c .

From now on, without loss the generality, we assume that $|M| = m = \sum_{c \in [k]} ns_c$. This implies that, in every feasible allocation A , we have $|A_i| = m/n$ for all $i \in [n]$.

4.2 Maximizing a Weighted Sum of Utilities

Our algorithm is based on searching the space of PO allocations. Particularly, we consider allocations that maximize a weighted sum of utilities $w_1u_1 + w_2u_2 + \dots + w_nu_n$, where each agent i associated with a weight $w_i \in [0, 1]$. Such allocations can be found by solving a maximum-weight matching problem in a weighted bipartite graph. We denote the set of all agents' weights by $w = (w_1, w_2, \dots, w_n)$.

Definition 4.1. For any n real numbers (weights) $w = (w_1, w_2, \dots, w_n)$, such that, $\forall i \in [n], w_i \in [0, 1]$, and $w_1 + w_2 + \dots + w_n = 1$, let G_w be a bipartite graph $(V_1 \cup V_2, E)$ with $|V_1| = |V_2| = m$. V_2 contains all m items (of all categories, include dummies). V_1 contains $\frac{m}{n}$ copies of each agent $i \in [n]$.

For each category $c \in [k]$, we choose distinct s_c copies of each agent and add an undirected edge from each of them to all the ns_c items of C_c . Each edge $\{i, o\} \in E, i \in V_1, o \in V_2$ has a weight $w(i, o)$, where:

$$w(i, o) := w_i \cdot u_i(o).$$

An allocation is called w -maximal if it corresponds to a maximum-weight matching among the maximum-cardinality matchings in G_w .

Proposition 4.2. Every w -maximal allocation, where $w_1, w_2, \dots, w_n \in (0, 1)$, is PO.

Proof. Every w -maximal allocation $A = (A_1, A_2, \dots, A_n)$ maximizes the sum $w_1u_1(A_1) + w_2u_2(A_2) + \dots + w_nu_n(A_n)$. Every Pareto-improvement would increase this sum. Therefore, there can be no Pareto-improvement, so A is PO. \square

4.3 Exchanging Pairs of Items

Our algorithm starts with a w -maximal allocation, and repeatedly exchanges pairs of items between the agents in order to find an allocation that is also EF1 (or EF[1,1]). To determine which pairs to exchange, we need some definitions and lemmas.

Definition 4.3. Given a feasible allocation $A = (A_1, A_2, \dots, A_n)$, an *exchangeable pair* is a pair (o_i, o_j) of items, $o_i \in A_i$ and $o_j \in A_j, i, j \in [n], i \neq j$, such that $A_i \setminus \{o_i\} \cup \{o_j\}$ and $A_j \setminus \{o_j\} \cup \{o_i\}$ are both feasible (equivalently: o_i and o_j are in the same category).

Note that, in a same-sign instance, for each agent, o_i, o_j are in the same "type", that is, both goods or both chores.

The following two lemmas deal with fairness while exchanging exchangeable pairs in a w -maximal allocation.

Lemma 4.4. Let A be a w -maximal feasible allocation, and let A' be another feasible allocation, resulting from A by exchanging an exchangeable pair (o_i, o_j) between some two agents $i \neq j$. Then there exists some ordering of the agents, k_1, \dots, k_n , such that for all $y > x$, the EF[1,1] condition is satisfied for agent k_y with respect to agent k_x in both allocations A and A' .

In particular, there is at least one agent (agent k_n) for whom both A and A' are EF[1,1]. In a same-sign instance, both A and A' are EF1 for agent k_n .

Proof. Let $A = (A_1, \dots, A_n)$ and $A' = (A'_1, \dots, A'_n)$. Let C_c be the category that contains both items o_i, o_j . By the pre-processing step, every bundle in A contains at least one item from C_c .

So we can write every bundle A_x , for all $x \in [n]$, as: $A_x = B_x \cup \{o_x\}$ for some $o_x \in C_c$, and $\forall x \neq i, j: A'_x = A_x = B_x \cup \{o_x\}$, and $A'_i = B_i \cup \{o_j\}$, $A'_j = B_j \cup \{o_i\}$.

Consider the envy-graph representing the partial allocation (B_1, B_2, \dots, B_n) . We claim that it contains no cycle. Suppose that it contained an envy-cycle. If we replaced the bundles according to the direction of edges in the cycle, we would get another feasible allocation which is a Pareto-improvement of the current allocation. Contradiction!

Therefore, the envy-graph of (B_1, B_2, \dots, B_n) has a topological ordering. Let k_1, \dots, k_n be such an ordering, so that for all $y > x$, agent k_y prefers B_{k_y} to B_{k_x} . In both allocations A and A' , the bundles of both k_y and k_x are derived from B_{k_y} and B_{k_x} by adding a single good or chore. Therefore, in both A and A' , the EF[1,1] condition is satisfied for agent k_y w.r.t. agent k_x . In particular, for agent k_n , both these allocations are EF[1,1].²

We now consider a same-sign instance. If C_c is a category of goods for agent k_n , then this implies $u_{k_n}(A_{k_n}) \geq u_{k_n}(B_y)$ and $u_{k_n}(A'_{k_n}) \geq u_{k_n}(B_y)$ for all $y \in [n]$, so both allocations are EF1 for agent k_n . If C_c is a category of chores for agent k_n , then this implies $u_x(B_{k_n}) \geq u_{k_n}(A_y)$ and $u_x(B_{k_n}) \geq u_{k_n}(A'_y)$, so again both allocations are EF1 for agent k_n . \square

Lemma 4.4 considered a single exchange. Now, we consider a sequence of exchanges. The following lemma works only for two agents — we could not yet extend it to more than two agents.

Lemma 4.5. *Suppose there are $n = 2$ agents. Suppose there is a sequence of feasible allocations A^1, \dots, A^k with the following properties:*

- *For every $j \in [k]$, the allocation $A^j = (A^j_1, A^j_2)$ is w -maximal, where $w = (w_{1,j}, w_{2,j})$ for some constants $w_{1,j}, w_{2,j} \in (0, 1)$.*
- *A^1 is envy-free for agent 1 and A^k is envy-free for agent 2.*
- *For every $j \in [k - 1]$, A^{j+1} is obtained from A^j by a single exchange of an exchangeable pair of items between the agents.*

Then, for some $j \in [k]$, the allocation A^j is PO and EF[1,1]. In a same-sign instance, the allocation A^j is PO and EF1.

Proof. Every A^j is PO by Proposition 4.2. Therefore, it is never possible for the two agents to envy each other simultaneously. Since at A^1 agent 1 is not envious and at A^k agent 2 is not envious, there must be some $j \in [k - 1]$ in which A^j is EF for 1, and A^{j+1} is EF for 2.

Because A^{j+1} results from A^j by exchanging an exchangeable pair between the agents, by Lemma 4.4, there exists an agent $i \in [2]$ such that both A^j and A^{j+1} are EF[1,1] for i .

If both are EF[1,1] for agent 1, then A^{j+1} is an EF[1,1] allocation. If both are EF[1,1] for agent 2, then A^j is an EF[1,1] allocation.

In a same-sign instance, all the above claims hold with EF1 instead of EF[1,1]. \square

To apply Lemma 4.5, we need a way to choose the pair of exchangeable items in each step of the sequence, so that the next allocation in the sequence remains w -maximal. We use the following definition.

²In fact, the result holds not only for an exchange of two items, but also for any permutation of n items of the same category, one item per agent. The proof is the same.

Definition 4.6. For a pair of agents $i, j, i \neq j$, and a pair of items (o_i, o_j) , the difference ratio is denoted by $r(i, j, o_i, o_j)$, and is equal to:

$$r(i, j, o_i, o_j) := \frac{u_j(o_i) - u_j(o_j)}{u_i(o_i) - u_i(o_j)}$$

If $u_j(o_i) = u_j(o_j)$ the ratio is always 0. If $u_i(o_i) = u_i(o_j)$ (and $u_j(o_i) \neq u_j(o_j)$), then the ratio is defined as $+\infty$ if $u_j(o_i) > u_j(o_j)$, or $-\infty$ if $u_j(o_i) < u_j(o_j)$.

Note that r is symmetric in the items, i.e. $r(i, j, o_i, o_j) = r(i, j, o_j, o_i)$.

In the case of two agents, i, j are always 1, 2. For simplicity, in that case, we denote $r(1, 2, o_1, o_2)$ by $r(o_1, o_2)$.

The following three lemmas show the properties of exchangeable pairs in a w -maximal allocation, and also the way to exchange pairs in order to preserve the PO.

Lemma 4.7. For any $w_1, w_2, \dots, w_n \in (0, 1)$ and allocation $A = (A_1, A_2, \dots, A_n)$, the following are equivalent:

(i) A is w -maximal.

(ii) For each $i, j \in [n], i \neq j$, for any exchangeable pair $o_i \in A_i, o_j \in A_j$,

$$w_i u_i(o_i) - w_j u_j(o_i) \geq w_i u_i(o_j) - w_j u_j(o_j) \quad (1)$$

$$w_i [u_i(o_i) - u_i(o_j)] \geq w_j [u_j(o_i) - u_j(o_j)] \quad (2)$$

(iii) For each $i, j \in [n], i \neq j$, for any exchangeable pair $o_i \in A_i, o_j \in A_j$,

$$u_i(o_i) > u_i(o_j) \quad \text{and} \quad w_i/w_j \geq r(i, j, o_i, o_j) \quad \text{or}$$

$$u_i(o_i) = u_i(o_j) \quad \text{and} \quad u_j(o_j) \geq u_j(o_i) \quad \text{or}$$

$$u_i(o_i) < u_i(o_j) \quad \text{and} \quad w_i/w_j \leq r(i, j, o_i, o_j).$$

Proof.

(i) \implies (ii): If an allocation is w -maximal, then its sum $w_1 u_1 + w_2 u_2 + \dots + w_n u_n$ should be, $\forall i, j \in [n], \forall o_i \in A_i, o_j \in A_j$, at least as large as in the allocation in which o_i and o_j are exchanged. So:

$$\begin{aligned} w_i u_i(o_i) + w_j u_j(o_j) &\geq w_i u_i(o_j) + w_j u_j(o_i) \\ \implies w_i u_i(o_i) - w_j u_j(o_i) &\geq w_i u_i(o_j) - w_j u_j(o_j) \end{aligned}$$

which is exactly (1). Equation (2) is derived from it by algebra.

(ii) \implies (i): Let $A' = (A'_1, A'_2, \dots, A'_n)$ be any other feasible allocation. Since $\forall i, j \in [n], A_i, A_j, A'_i, A'_j$ all have the same cardinality, there must be a sequence of pairs of exchangeable items $\{(o_{i_k}^k, o_{j_k}^k)\}_{k=1,2,\dots}$ such that, for all k , i_k and j_k are some agents, and when these pairs are switched, the allocation changes from A to A' . By (1), for each such pair k ,

$$\begin{aligned} w_i u_i(o_{i_k}^k) - w_j u_j(o_{i_k}^k) &\geq w_i u_i(o_{j_k}^k) - w_j u_j(o_{j_k}^k) \\ \implies w_i u_i(o_{i_k}^k) + w_j u_j(o_{j_k}^k) &\geq w_i u_i(o_{j_k}^k) + w_j u_j(o_{i_k}^k). \end{aligned}$$

Therefore, when pair k is switched, the sum $w_{i_k} u_{i_k} + w_{j_k} u_{j_k}$ weakly decreases. Since the bundles of agents not in $\{i_k, j_k\}$ do not change, the total weighted sum $w_1 u_1 + \dots + w_n u_n$ weakly decreases.

Since it holds for each k , the sum $w_1u_1 + \dots + w_nu_n$ in A is weakly larger than in A' . This holds for all allocations A' ; so A is w -maximal.

(ii) \iff (iii): The claim in (iii) is an algebraic manipulation of (2). \square

Lemma 4.8. *In any w -maximal allocation A , for any i, j and an exchangeable pair (o_i, o_j) with $o_i \in A_i, o_j \in A_j$, the following implications hold:*

$$\begin{aligned} u_j(o_i) \geq u_j(o_j) &\implies u_i(o_i) \geq u_i(o_j) \\ u_j(o_i) > u_j(o_j) &\implies u_i(o_i) > u_i(o_j) \end{aligned}$$

Proof. Follows immediately from 4.7 (2). \square

Definition 4.9. Consider a w -maximal allocation A and an exchangeable pair (o_i, o_j) , such that $o_i \in A_i, o_j \in A_j$, for some $i, j \in [n]$. o_i is called a *preferred item* in the exchangeable pair (o_i, o_j) if $u_j(o_i) > u_j(o_j)$ (which implies $u_i(o_i) > u_i(o_j)$ by Lemma 4.8).

Lemma 4.10. *In any w -maximal allocation A , if an agent j envies some agent i , then there is an exchangeable pair (o_i, o_j) , such that $o_i \in A_i, o_j \in A_j$, and o_i is the preferred item.*

Proof. If j envies i , then $u_j(A_i) > u_j(A_j)$. Since both A_i and A_j contain the same number of items in each category, there must be a category in which, for some item pair $o_i \in A_i$ and $o_j \in A_j$, agent j prefers o_i to o_j . \square

The following lemma shows that, by exchanging items, we can move from one w -maximal allocation to another w' -maximal allocation. This lemma, too, works only for two agents.

Lemma 4.11. *Suppose there are $n = 2$ agents. Let A be a w -maximal allocation, for $w = (w_1, w_2)$. Suppose there is an exchangeable pair $o_1 \in A_1, o_2 \in A_2$ such that:*

1. $u_2(o_1) > u_2(o_2)$, that is, o_1 is the preferred item.
2. Among all exchangeable pairs with a preferred item, this pair has the largest difference-ratio $r(o_1, o_2)$.

Let A' be the allocation resulting from exchanging o_1 and o_2 in A . Then, A' is w' -maximal for some $w' = (w'_1, w'_2)$ with $w'_1 \leq w_1, w'_2 \geq w_2, w'_1 \in (0, 1), w'_2 \in (0, 1)$.

Proof. The lemma can be proved by using Lemmas 4.7, 4.8, the maximality condition [condition 2] and Definition 4.6.

The idea of the proof is to define $w'_1, w'_2 \in (0, 1)$ such that $\frac{w'_1}{w'_2} = r(o_1, o_2), w'_1 + w'_2 = 1$. Then, $0 < \frac{w'_1}{w'_2} \leq \frac{w_1}{w_2}$, and $w'_1 \leq w_1, w'_2 \geq w_2$.

Then we look at all the exchangeable pairs (o_1^*, o_2^*) in the new allocation A' , resulting from the exchange, and see that they satisfy all the conditions of Lemma 4.7(iii) with w'_1, w'_2 , which are:

- (a) $u_1(o_1^*) > u_1(o_2^*)$ and $r(o_1, o_2) \geq r(o_1^*, o_2^*)$ or
- (b) $u_1(o_1^*) = u_1(o_2^*)$ and $u_2(o_2^*) \geq u_2(o_1^*)$ or
- (c) $u_1(o_1^*) < u_1(o_2^*)$ and $r(o_1, o_2) \leq r(o_1^*, o_2^*)$

The exchangeable pairs can be divided into four types:

1. The exchangeable pairs (o_1^*, o_2^*) that have not moved.
2. The pair (o_2, o_1) .
3. Pairs in the form (o_1^*, o_1) , $o_1^* \in A'_1$, $o_1^* \neq o_2$.
4. Pairs in the form (o_2, o_2^*) , $o_2^* \in A'_2$, $o_2^* \neq o_1$.

We show that each pair of each type satisfies its own condition out of (a), (b) and (c). Therefore, by Lemma 4.7 (i, iii), A' is w -maximal allocation, for (w'_1, w'_2) .

The complete proof with all the technical arguments can be found in the Appendix. \square

To apply Lemma 4.5, we also need a w -maximal allocation that is envy-free for agent 1 (and another one that is envy-free for agent 2). If $w_1 = 1$, then obviously a w -maximal allocation is envy-free for agent 1, since it maximizes the utility of agent 1. But, in this case $w_2 = 0$, so Proposition 4.2 does not work. Therefore, we must show that there exists a w -maximal allocation that is envy-free for agent 1, where both weights are positive (and the same for agent 2).

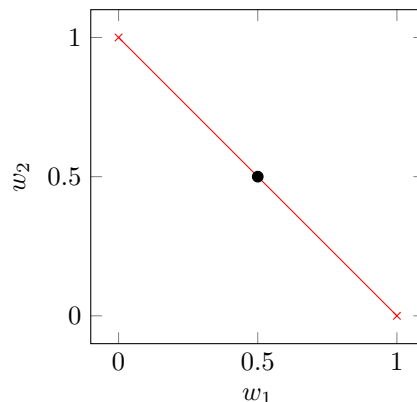
Lemma 4.12. *Suppose there are $n = 2$ agents. For each agent $j \in [n]$, there exists an allocation that is w -maximal for some w in which all weights are positive, and it is envy-free for agent j .*

Proof. Start with a w -maximal allocation for $w = (0.5, 0.5)$. If it is envy-free for agent j , then we are done. Otherwise, by Lemma 4.10, there is an exchangeable pair with a preferred item. Exchanging this pair leads, by Lemma 4.11, to a w' -maximal allocation in which the weights in w' are strictly positive. If j is still envious, repeat the argument. Each exchange strictly increases the utility of agent j . Therefore, after finitely many exchanges, the allocation is envy-free for j . \square

4.4 Algorithm Description

Throughout this section we consider same-sign instances, for simplicity. All the results are valid for general mixed instances, by replacing EF1 with EF[1,1].

Let us start with an intuitive description of the algorithm, for two agents. Suppose that w_2 is a function of w_1 , and consider the line $w_1 + w_2 = 1$, $w_1 \geq 0$, $w_2 \geq 0$, which describes the collection of all pairs of non-negative weights $w_1, w_2 \in (0, 1)$ whose sum is 1:



	o_1	o_2	o_3	o_4	o_5	o_6
Agent 1	0	-1	-4	-5	0	2
Agent 2	0	-1	-2	-1	-1	0

Table 1: Utilities of the agents in the example.

The algorithm begins to divide the items by a maximum weighted matching in the graph G_w , where $w = (0.5, 0.5)$.

Note that at each point on this line, the corresponding allocation is w -maximal. In particular, there are no envy-cycles in the envy graph, so there exist at most one envious agent.

According to that, the initial allocation is PO (By Lemma 4.2) and envy-free for at least one agent. If it is envy-free for both agents then we are done. Otherwise, Depending on the envious agent, the algorithm decides which side of the line to go to. If agent 2 envies, we need to improve 2's weight, so we go towards $(0,1)$. If agent 1 envies we need to go towards $(1,0)$. Therefore, as long as the allocation is not EF1, the algorithm swaps an exchangeable pair chosen according to Lemma 4.11, thus maintaining the search space as the space of the w -maximal allocations. Note that since the items of the exchanged pair are both in the same category, the capacity constraints are also maintained. Lemma 4.5 implies that some point on the line gives a feasible EF1 and PO division.

Specifically, the exchange points are determined as follows. For each item o we can define a linear function $f(w_1)$:

$$\begin{aligned}
 w_1 u_1(o) - w_2 u_2(o) &= w_1 u_1(o) - (1 - w_1) u_2(o) \\
 &= w_1 u_1(o) - u_2(o) + w_1 u_2(o) \\
 &= (u_1(o) + u_2(o)) w_1 - u_2(o)
 \end{aligned}$$

If we draw all those functions in one coordinate system, each pair of lines intersect at most once. In total there are $O(m^2)$ intersections, where $m = \sum_{c \in [k]} |C_c|$, the total number of items, in all categories (including the dummies).

For example, consider the instance $I = (N, M, C, S, U)$ where $N = [2]$, $C = \{C_1, C_2\}$, $C_1 = \{o_1, o_2, o_3, o_4\}$, $C_2 = \{o_5, o_6\}$, $S = \{2, 1\}$ and U is shown in Table 1. The corresponding lines for the items are depicted in Figure 1.

The meaning of each point of intersection is a possible switching point for these two items between the agents. Clearly, the replacement will only take place between exchangeable pairs, i.e. items in the same category, which are in different agents' bundles at the time of the intersection.

According to Definition 4.6, at each intersection point of the lines of o_1 and o_2 , $\frac{w_1}{w_2} = \frac{u_2(o_1) - u_2(o_2)}{u_1(o_1) - u_1(o_2)} = r(o_1, o_2)$ holds. The largest r value is obtained on the right side of the graph, and as we progress to the left side its value decreases.

In this example, the algorithm starts with the allocation $A = (A_1, A_2)$ in the point $(0.5, 0.5)$, which is $A_1 = \{o_1, o_2, o_6\}$, $A_2 = \{o_3, o_4, o_5\}$. Note that for each category, 1's items are the top lines. In this initial allocation, 2 envies by more than one item, so we start exchanging items in order to increase w_2 . The first intersecting pair (when we go left) is o_5, o_6 . It is an exchangeable pair, so we exchange it and update the allocation to $A_1 = \{o_1, o_2, o_5\}$, $A_2 = \{o_3, o_4, o_6\}$. This is an EF1 allocation, so we are done.

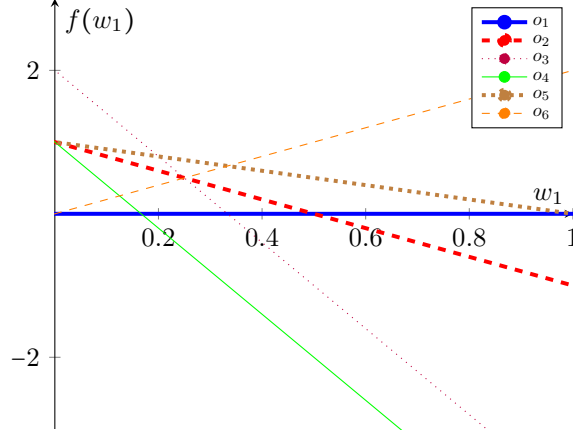


Figure 1: The corresponding lines for the items in the example.

Algorithm 1 Finding an EF1 and PO division for 2 agents

```

// Step 1: Find a  $w$ -maximal feasible allocation that is EF for some agent.
1:  $A = (A_1, A_2) \leftarrow$  a  $w$ -maximal allocation, for  $w_1 = w_2 = 0.5$ .
2: if  $A$  is EF1 then
3:   return  $A$ 
4: end if
5: if  $A$  is EF for agent 2 then
6:   replace the names of agent 1 and agent 2
7: end if
// We can now assume that agent 2 is jealous.
// Step 2: Build a set of item-pairs whose replacement increases agent 2's utility:
8: item-pairs  $\leftarrow$  all the exchangeable pairs  $(o_1, o_2)$  for which  $o_1 \in A_1, o_2 \in A_2, u_2(o_1) > u_2(o_2)$ .
9: current-pair  $\leftarrow (o_1, o_2)$  where  $r(o_1, o_2)$  is maximal.
// Step 3: Switch items in order until an EF1 allocation is found:
10: while  $A = (A_1, A_2)$  is not EF1 do
11:   Switch current-pair between the agents.
12:   Update item-pairs list and current-pair (Steps 8, 9).
13: end while
14: return  $A$ 
    
```

If at some point there are multiple intersections of exchangeable pairs, we swap the pairs in an arbitrary order.

Lemma 4.13. *If the algorithm exchanges the last exchangeable pair in the list, then the resulting allocation is envy-free for agent 2.*

Proof. After the last exchange, there is no exchangeable pair (o_1, o_2) for which o_1 is the preferred item. Therefore, by Lemma 4.10, agent 2 is not envious. \square

Theorem 4.14. *Algorithm 1 always returns an allocation that is PO and satisfies the capacity constraints. The allocation is EF1 for a same-sign instance, and EF[1,1] for a general mixed instance.*

Proof. A matching in G_w graph always gives each agent s_c items of category C_c . Thanks to the dummy items, all possible allocations that satisfy the capacity constraints can be obtained by a matching. The first allocation that the algorithm checks is some w -maximal allocation, where $w = (w_1, w_2), w_1, w_2 \in (0, 1)$, so by Proposition 4.2, this is a PO allocation.

At each iteration, it exchanges an exchangeable pair, (o_1, o_2) , such that $u_2(o_1) > u_2(o_2)$, and among all the exchangeable pairs with $u_2(o_1) > u_2(o_2)$ it has the largest $r(o_1, o_2)$, so by Lemma 4.11, the resulting allocation is also w' -maximal for some $w' = (w'_1, w'_2)$. In addition, since the items are in the same category, the allocation remains feasible.

The first allocation in the sequence is, by step 1, envy-free for agent 1. By Lemma 4.13, the last allocation in the sequence is envy-free for agent 2. So by Lemma 4.5, there exists some iteration in which the allocation is PO and EF1 (for a same-sign instance) or EF[1,1] (for a general mixed instance). \square

Theorem 4.15. *The runtime of the algorithm is $O(m^4 \log(m^2))$.*

Proof. Step 1 can be done by finding a maximum weighted matching in a bipartite graph G_w . Its time complexity is $O(|V|^3)$, where $|V| = 2m$, the number of vertices in the graph. Thus, $O(m^3)$ is the time complexity of step 1.

At step 2 we go through all the categories $c \in [k]$, where in each category we create groups $A_{1,c}, A_{2,c}$ which contains agent 1's and agent 2's items from C_c in A . It can be done in $\frac{m}{2}|C_c| = ms_c$. Now we have $|A_{1,c}| = |A_{2,c}| = s_c$. Then, we iterate over all the pairs $(o_1, o_2) \in (A_{1,c}, A_{2,c})$, and add them to the list, which takes s_c^2 time. In total, building item-pairs list is $\sum_{c \in [k]} (ms_c + s_c^2) = O(\sum_{c \in [k]} ms_c) = O(km^2)$. The item-pairs list size is $\sum_{c \in [k]} s_c^2 = O(m^2)$, and then finding its maximum takes $O(m^2)$. In total, step 2 takes $O(km^2)$ time.

The upper bound on the number of iterations in the while loop at step 3 is the number of intersection points between items, which is at most $O(m^2)$. At each iteration we switch one exchangeable pair, (o_1, o_2) , and update the pairs-list. The only pairs that should be updated (deleted or added) are those that contain o_1 or o_2 . There are at most $2m = O(m)$ such pairs. Finding the maximum is $O(m^2)$. In total, step 3 takes $O(m^4)$ time.

Overall, the time complexity of the algorithm is $O(m^4)$ (because $m \geq k$ necessarily). \square

5 Conclusion and Future Work

We presented the first algorithm for efficient nearly-fair allocation of mixed goods and chores with capacity constraints. Our proofs are modular, and some of our lemmas can be used in more general settings.

5.1 Three or more agents

The most interesting challenge is to generalize our algorithm to three or more agents. Lemmas 4.4, 4.7, 4.8, 4.10 work for any number of agents, but the other lemmas currently work only for two agents.

Algorithm 1 essentially scans the space of w -maximal allocations: it starts with one w -maximal allocation, and then moves in the direction that increases the utility of the envious agent. To extend it to n agents, we can similarly start with a w -maximal allocation corresponding to $w = (1/n, \dots, 1/n)$. Then, for every pair of agents i, j such that i envies j , we can find a exchangeable pair between i and j with a maximal ratio. We can then try all these exchanges and thus search the space of allocations in a breadth-first-search manner. We do not know whether this approach will always find an EF[1,1] allocation.

5.2 More general constraints

Another possible generalization is to more general constraints, such as laminar matroids, base-orderable matroids, or general matroids. Lemmas 4.2, 4.4, 4.5, 4.8 and 4.11 do not use categories, and should work for general matroids. The other lemmas should be adapted.

Finally, we assumed that both agents have the same capacity constraints. We do not know if our results can be extended to agents with different capacity constraints (e.g. agent 1 can get at most 7 items while agent 2 can get at most 3 items). Specifically, the proof of Lemma 4.4 does not work — if (A_1, A_2) is feasible, then (A_2, A_1) might be infeasible.

Acknowledgment

This research was supported in part by the Ministry of Science, Technology & Space of Israel, and by the Israel Science Foundation (grant no. 712/20).

References

- Aleksandrov, M. and Walsh, T. (2019). Greedy algorithms for fair division of mixed manna. *arXiv preprint arXiv:1911.11005*.
- Aziz, H., Caragiannis, I., Igarashi, A., and Walsh, T. (2022). Fair allocation of indivisible goods and chores. *Autonomous Agents and Multi-Agent Systems*, 36(1):1–21.
- Aziz, H., Moulin, H., and Sandomirskiy, F. (2020). A polynomial-time algorithm for computing a pareto optimal and almost proportional allocation. *Operations Research Letters*, 48(5):573–578.
- Barman, S., Krishnamurthy, S. K., and Vaish, R. (2018a). Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 557–574.
- Barman, S., Krishnamurthy, S. K., and Vaish, R. (2018b). Finding fair and efficient allocations. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, pages 557–574.
- Bérczi, K., Bérczi-Kovács, E. R., Boros, E., Gedefa, F. T., Kamiyama, N., Kavitha, T., Kobayashi, Y., and Makino, K. (2020). Envy-free relaxations for goods, chores, and mixed items. *arXiv preprint arXiv:2006.04428*.
- Bhaskar, U., Sricharan, A., and Vaish, R. (2021). On approximate envy-freeness for indivisible chores and mixed resources. *arXiv preprint arXiv:2012.06788*.
- Bilò, V., Caragiannis, I., Flammini, M., Igarashi, A., Monaco, G., Peters, D., Vinci, C., and Zwicker, W. S. (2022). Almost envy-free allocations with connected bundles. *Games and Economic Behavior*, 131:197–221.
- Biswas, A. and Barman, S. (2018). Fair division under cardinality constraints. In *Proceedings of the Twenty-Seventh International Joint Conference on Artificial Intelligence*, pages 91–97.
- Bouveret, S., Chevaleyre, Y., and Maudet, N. (2016). Fair allocation of indivisible goods. In Brandt, F., Conitzer, V., Endriss, U., Lang, J., and Procaccia, A. D., editors, *Handbook of Computational Social Choice*, pages 284–310. Cambridge University Press.
- Brams, S. J. (2007). *Mathematics and democracy: Designing better voting and fair-division procedures*. Princeton University Press.
- Brams, S. J. and Taylor, A. D. (1996). *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press.
- Brustle, J., Dippel, J., Narayan, V. V., Suzuki, M., and Vetta, A. (2020). One dollar each eliminates envy. In *Proceedings of the 21st ACM Conference on Economics and Computation*, page 23–39.
- Budish, E. (2011). The combinatorial assignment problem: Approximate competitive equilibrium from equal incomes. *Journal of Political Economy*, 119(6):1061–1103.
- Caragiannis, I., Kurokawa, D., Moulin, H., Procaccia, A. D., Shah, N., and Wang, J. (2019). The unreasonable fairness of maximum nash welfare. *ACM Transactions on Economics and Computation (TEAC)*, 7(3):1–32.
- Chen, X. and Liu, Z. (2020). The fairness of leximin in allocation of indivisible chores. *arXiv preprint arXiv:2005.04864*.
- Dror, A., Feldman, M., and Segal-Halevi, E. (2021). On fair division under heterogeneous matroid constraints. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 5312–5320.
- Gafni, Y., Huang, X., Lavi, R., and Talgam-Cohen, I. (2021). Unified fair allocation of goods and chores via copies. *arXiv preprint arXiv:2109.08671*.

- Hummel, H. and Hetland, M. L. (2021). Guaranteeing half-maximin shares under cardinality constraints. *arXiv preprint arXiv:2106.07300*.
- Igarashi, A. and Peters, D. (2019). Pareto-optimal allocation of indivisible goods with connectivity constraints. In *Proceedings of the AAAI conference on artificial intelligence*, pages 2045–2052.
- Mackin, E. and Xia, L. (2016). Allocating indivisible items in categorized domains. In *Proceedings of the Twenty-Fifth International Joint Conference on Artificial Intelligence*, pages 359–365.
- Moulin, H. (2004). *Fair division and collective welfare*. MIT press.
- Negishi, T. (1960). Welfare economics and existence of an equilibrium for a competitive economy. *Metroeconomica*, 12(2-3):92–97.
- Nyman, K., Su, F. E., and Zerbib, S. (2020). Fair division with multiple pieces. *Discrete Applied Mathematics*, 283:115–122.
- Sikdar, S., Adali, S., and Xia, L. (2017). Mechanism design for multi-type housing markets. In *Proceedings of the Thirty-First AAAI Conference on Artificial Intelligence*, pages 684–690.
- Sikdar, S., Adali, S., and Xia, L. (2019). Mechanism design for multi-type housing markets with acceptable bundles. In *Proceedings of the AAAI Conference on Artificial Intelligence*, pages 2165–2172.
- Suksompong, W. (2021). Constraints in fair division. *ACM SIGecom Exchanges*, 19(2):46–61.
- Varian, H. R. (1976). Two problems in the theory of fairness. *Journal of Public Economics*, 5(3-4):249–260.
- Weller, D. (1985). Fair division of a measurable space. *Journal of Mathematical Economics*, 14(1):5–17.
- Wu, X., Li, B., and Gan, J. (2020). Budget-feasible maximum nash social welfare allocation is almost envy-free. *arXiv preprint arXiv:2012.03766*.

Appendix

A Methods that Do Not Work

In this section, we present some of our attempts to find an EF1 and PO allocation, using ideas from previous works. These attempts failed, and it shows that the problem is not trivial, and the new tools that have been developed in this paper are required.

A.1 Iterated Matching Algorithm

The Iterated Matching algorithm, presented by Brustle et al. (2020), finds an EF1 allocation of indivisible goods in the case of additive valuations. This is done by using the *valuation graph*, which is a complete bipartite graph on vertex sets I (n agents) and J (m goods), with weights that representing the agents' utilities. The algorithm proceeds in rounds where each agent is matched to exactly one item in each round, by a maximum weighted matching on a sub graph of all the remaining goods, until all items have been allocated.

This algorithm can be easily applied to chores with one category, by adding at the beginning some k dummy chores (with utility of 0 to each agent), where $|J| = an - k$, and $a \in \mathbb{N}, k \in \{0, \dots, n - 1\}$.

However, with more than one category, we need to add an external loop that runs on all categories, and at each iteration executes the algorithm for chores. While it maintains capacity constraints (because in each iteration all the agents get chores, similarly to round robin), it may not necessarily maintain the EF1 requirement, as we show in the following example. Denote by $o_{i,j}$ the j -th item of category i .

	$o_{1,1}$	$o_{1,2}$	$o_{2,1}$	$o_{2,2}$
Agent 1	0	-2	-2	-1
Agent 2	0	-4	-4	0

After allocating category C_1 , $A_1 = \{o_{1,2}\}, A_2 = \{o_{1,1}\}$, so $u_1(A_1) = -2, u_1(A_2) = 0, u_2(A_1) = -4, u_2(A_2) = 0$, then agent 1 envies 2, but up to one item (the only item), and agent 2 is not envious.

Then we allocate the second category, which changes the allocations to: $A_1 = \{o_{1,2}, o_{2,1}\}, A_2 = \{o_{1,1}, o_{2,2}\}$. Now $u_1(A_1) = -4, u_1(A_2) = -1$, so agent 1 envies in more than one item (her worst chore has utility of -2).

If there was always an agent who was not jealous, we could have assigned her the new chore, but this is not guaranteed. An envy-cycle may be created, and we know that envy-cycle elimination algorithm may fail EF1 for additive chores (Bhaskar et al., 2021).

A.2 Top-trading Envy Cycle Elimination Algorithm

Bhaskar et al. (2021) considered fair allocation of chores, and suggested to use cycle elimination on the *top-trading graph*, instead of the usual envy-graph. The top-trading graph for a division A is a directed graph on the vertex set N , with a directed edge from agent i to agent k if $u_i(A_k) = \max_{j \in N} u_i(A_j)$ and $u_i(A_k) > u_i(A_i)$, i.e. A_k is the most preferred bundle for agent i in A , and she prefers A_k over her own bundle. They showed that resolving a top-trading envy cycle preserves EF1. Indeed, every agent involved in the top-trading exchange receives its most preferred bundle

after the swap, and therefore does not envy anyone else in the next round. They also define what a "sink" agent is: an agent on the top-trading graph called "sink" if there are no arcs that come out of it, i.e. does not envy nobody. In addition, they proved that if the usual envy-graph does not have a "sink", then the top-trading envy graph has a cycle.

In their algorithm, for each chore, they construct the envy-graph. If there is no sink in it, they eliminate cycles on the top-trading envy-graph, which guarantees the existence of a sink agent in the envy graph, and then allocate the chore to a sink agent.

This method does not work in the setting with capacity constraints, because we can not simply assign the new chore (or the item) to the sink agent, because she may already have the maximum allowed number of chores from this category.

For example, consider the following instance with capacity constraints $S = \{2, 1\}$:

	$o_{1,1}$	$o_{1,2}$	$o_{1,3}$	$o_{1,4}$	$o_{2,1}$	$o_{2,2}$
Agent 1	-1	0	0	0	-2	-4
Agent 2	-1	0	0	0	-1	-3

At the beginning of the algorithm, both allocations are empty, so agent 1 and agent 2 are both sinks. Say that agent 1 was selected to get $o_{1,1}$, and now $A_1 = \{o_{1,1}\}$, $A_2 = \emptyset$. Now agent 1 envious, so the only sink agent is 2, and $o_{1,2}$ is allocated to 2. Since $u_2(o_{1,2}) = u_2(o_{1,3}) = 0$, agent 2 stays sink in the two following iterations. Then, the new allocations are $A_1 = \{o_{1,1}\}$, $A_2 = \{o_{1,2}, o_{1,3}\}$, and the only sink is agent 2. According to the algorithm, we should assign $o_{1,4}$ to agent 2, but then we exceed capacity constraints.

A.3 Greedy Round-robin with Cycle Elimination

Biswas and Barman (2018) solved the problem of allocating goods under capacity constraints. Their algorithm first determines an arbitrary ordering of the n agents, σ , and then for each category: uses the Greedy Round-Robin algorithm to allocate the goods of this category, eliminate the cycles on the envy-graph, and update σ to be a topological ordering of the envy-graph.

As already mentioned, this algorithm will not work for chores, because eliminating cycles in the usual envy-graph may violate EF1 (Bhaskar et al., 2021).

In addition, if we use the top-trading graph instead, we can not set the topological ordering according to it, but on the envy-graph. It is possible that the top-trading graph is cycle-free, while the envy-graph has cycles, even though it has no sinks. So we may not necessarily have a topological ordering on it.

A.4 Pareto-improvement of EF1 Allocation

We examined the approach of finding an EF1 allocation, which is not necessarily PO, and applying Pareto improvements to it until an EF1 and PO allocation is obtained. However, the following proposition shows that this approach is inadequate, even with two agents.

Proposition 1. Not every Pareto-improvement of an EF1 allocation yields an EF1 allocation, even with two agents.

Proof. Let $A = \{A_1, A_2\}$ be an EF1 allocation for two agents. Define a Pareto improvement as a replacement between two subsets of chores: $X_1 \subseteq A_1$ and $X_2 \subseteq A_2$ (one of the subsets may be

empty), such that the change harms no one and benefits at least one agent. In particular, $\forall i \in 1, 2 : u_i(X_{3-i}) \geq u_i(X_i)$ - the agent prefers (or indifferent) what she received over what she gave.

Let us see an example that proves the proposition. Consider an instance with one category with 8 chores, capacity constraint of 8, and two agents with the following valuations:

	1	2	3	4	5	6	7	8
Agent 1	-5	-2	-1	-2	-2	-2	-1	-2
Agent 2	-1	-1	-2	-1	-1	0	0	0

Suppose that the EF1 allocation, A , is $A_1 = \{1, 5, 6, 7\}$, $A_2 = \{2, 3, 4, 8\}$. The utilities of the agents in A are:

- $u_1(A_1) = -10, u_1(A_2) = -7$
- $u_2(A_1) = -2, u_2(A_2) = -4$

Clearly, the two agents are jealous of each other, but the envy is up to one chore because $u_1(A_1 \setminus \{1\}) \geq u_1(A_2)$, and $u_2(A_2 \setminus \{3\}) \geq u_2(A_1)$.

In addition, A is not PO because there is an envy-cycle in the envy-graph.

Consider the following Pareto-improvement: $X_1 = \{1\}, X_2 = \{2, 3, 4\}$. It does not harm agent 1 and benefits agent 2.

After the replacement, the utilities of agent 1 do not change, that is, $u_1(A_1) = -10, u_1(A_2) = -7$. However, the most difficult chore in agent 1's bundle is worth -2 , which is not enough for her to eliminate the envy. So the Pareto-improvement is not EF1. \square

B Technical Lemmas

Lemma B.1. *For any six real numbers $x_i, x_j, y_i, y_j, z_i, z_j$, the following inequalities are equivalent:*

$$x_i(y_j - z_j) + y_i(z_j - x_j) + z_i(x_j - y_j) \leq 0 \quad (3)$$

$$(x_i - y_i)(x_j - z_j) \leq (x_i - z_i)(x_j - y_j) \quad (4)$$

$$(y_i - z_i)(y_j - x_j) \leq (y_i - x_i)(y_j - z_j) \quad (5)$$

$$(z_i - x_i)(z_j - y_j) \leq (z_i - y_i)(z_j - x_j) \quad (6)$$

Proof. By adding $+x_i x_j - x_i x_j$ to inequality (3), we get:

$$\begin{aligned} & (3) \\ \iff & x_i(y_j - z_j + x_j - x_j) + y_i(z_j - x_j) + z_i(x_j - y_j) \leq 0 \\ \iff & x_i((x_j - z_j) - (x_j - y_j)) - y_i(x_j - z_j) + z_i(x_j - y_j) \leq 0 \\ \iff & (x_i - y_i)(x_j - z_j) - (x_i - z_i)(x_j - y_j) \leq 0 \\ \iff & (x_i - y_i)(x_j - z_j) \leq (x_i - z_i)(x_j - y_j) \\ \iff & (4). \end{aligned}$$

Inequalities (5), (6) are entirely analogous. \square

Lemma B.2. *If $x_i > y_i > z_i$, then the following are equivalent:*

$$\frac{x_j - z_j}{x_i - z_i} \leq \frac{x_j - y_j}{x_i - y_i} \quad r(i, j, x, z) \leq r(i, j, x, y) \quad (7)$$

$$\frac{y_j - z_j}{y_i - z_i} \leq \frac{y_j - x_j}{y_i - x_i} \quad r(i, j, y, z) \leq r(i, j, x, y) \quad (8)$$

$$\frac{z_j - y_j}{z_i - y_i} \leq \frac{z_j - x_j}{z_i - x_i} \quad r(i, j, y, z) \leq r(i, j, x, z) \quad (9)$$

Proof. For each k in $\{4,5,6\}$, divide inequality (k) by the two terms with the “ i ” subscript to get inequality ($k+3$). \square

Observation B.3. Lemma B.1 is still true if we reverse all inequalities directions, so too Lemma B.2.

That is, if $x_i > y_i > z_i$, then the following are equivalent:

$$\frac{x_j - z_j}{x_i - z_i} \geq \frac{x_j - y_j}{x_i - y_i} \quad r(i, j, x, z) \geq r(i, j, x, y) \quad (10)$$

$$\frac{y_j - z_j}{y_i - z_i} \geq \frac{y_j - x_j}{y_i - x_i} \quad r(i, j, y, z) \geq r(i, j, x, y) \quad (11)$$

$$\frac{z_j - y_j}{z_i - y_i} \geq \frac{z_j - x_j}{z_i - x_i} \quad r(i, j, y, z) \geq r(i, j, x, z) \quad (12)$$

C A Complete Proof for Lemma 4.11

Lemma 4.11. *Suppose there are $n = 2$ agents. Let A be a w -maximal allocation, for $w = (w_1, w_2)$. Suppose there is an exchangeable pair $o_1 \in A_1, o_2 \in A_2$ such that:*

1. $u_2(o_1) > u_2(o_2)$, that is, o_1 is the preferred item.
2. Among all exchangeable pairs with a preferred item, this pair has the largest difference-ratio $r(o_1, o_2)$.

Let A' be the allocation resulting from exchanging o_1 and o_2 in A . Then, A' is w' -maximal for some $w' = (w'_1, w'_2)$ with $w'_1 \leq w_1, w'_2 \geq w_2, w'_1 \in (0, 1), w'_2 \in (0, 1)$.

Proof. By Lemma 4.8, $u_1(o_1) > u_1(o_2)$, because A is w -maximal and (o_1, o_2) is an exchangeable pair for which $u_2(o_1) > u_2(o_2)$. Then, by definition, $r(o_1, o_2) > 0$, and by lemma 4.7(iii), $\frac{w_1}{w_2} \geq r(o_1, o_2)$.

Consider the allocation $A' = (A'_1, A'_2)$, resulting from exchanging o_1 and o_2 . Consider some $w'_1, w'_2 \in (0, 1)$ such that $\frac{w'_1}{w'_2} = r(o_1, o_2)$, so $0 < \frac{w'_1}{w'_2} \leq \frac{w_1}{w_2}$.

We now look at all the exchangeable pairs (o_1^*, o_2^*) after the exchange, and see that they satisfy all the conditions of Lemma 4.7(iii) with $w' = (w'_1, w'_2)$, which can be written as:

- (a) $u_1(o_1^*) > u_1(o_2^*)$ and $r(o_1, o_2) \geq r(o_1^*, o_2^*)$ or

- (b) $u_1(o_1^*) = u_1(o_2^*)$ and $u_2(o_2^*) \geq u_2(o_1^*)$ or
 (c) $u_1(o_1) < u_1(o_2)$ and $r(o_1, o_2) \leq r(o_1^*, o_2^*)$

1. The exchangeable pairs (o_1^*, o_2^*) who have not moved: Lemma 4.7 implies that:

- If $u_1(o_1^*) > u_1(o_2^*)$, then $\frac{w_1}{w_2} \geq r(o_1^*, o_2^*)$. If also $u_2(o_1^*) > u_2(o_2^*)$, then the maximality of $r(o_1, o_2)$ [condition 2] implies $\frac{w_1}{w_2} \geq r(o_1, o_2) \geq r(o_1^*, o_2^*)$. Else, by Definition 4.6, $r(o_1^*, o_2^*) \leq 0$, so $r(o_1, o_2) \geq 0 \geq r(o_1^*, o_2^*)$ holds again.
- If $u_1(o_1^*) = u_1(o_2^*)$, then $u_2(o_2^*) \geq u_2(o_1^*)$.
- If $u_1(o_1^*) < u_1(o_2^*)$, then $\frac{w_1}{w_2} \leq r(o_1^*, o_2^*)$. In particular $r(o_1, o_2) \leq r(o_1^*, o_2^*)$.

After the exchange, they still satisfy the same conditions.

2. The pair (o_2, o_1) :

Now $o_2 \in A'_1, o_1 \in A'_2$, and the pair (o_2, o_1) fits condition (c), which says that the item in 1's bundle worth less, for agent 1, than the item in 2's bundle ($u_1(o_2) < u_1(o_1)$). For it $\frac{w'_1}{w'_2} = r(o_1, o_2) = r(o_2, o_1)$, by definition and by symmetry of r .

3. Pairs in the form (o_1^*, o_1) , $o_1^* \in A'_1, o_1^* \neq o_2$:

- (a) If $u_1(o_1^*) > u_1(o_1)$, of course $u_1(o_1^*) > u_1(o_1) > u_1(o_2)$. If also $u_2(o_1^*) > u_2(o_1)$ then because of the maximality condition, $r(o_1, o_2) \geq r(o_1^*, o_2)$. By B.3 (with $x = o_1^*, y = o_1, z = o_2$), this is equivalent to $r(o_1, o_2) \geq r(o_1^*, o_1)$. And if $u_2(o_1^*) \leq u_2(o_1)$, then $r(o_1^*, o_1) \leq 0 < r(o_1, o_2)$.
- (b) If $u_1(o_1^*) = u_1(o_1)$, it is not possible that $u_2(o_1) < u_2(o_1^*)$ because it implies $u_2(o_2) < u_2(o_1^*)$, and by 4.8, $u_1(o_2) < u_1(o_1^*)$. By the values of r 's numerators and denominators, we get $r(o_1^*, o_2) > r(o_1, o_2)$, and it contradicts $r(o_1, o_2)$ maximality. Therefore, $u_2(o_1) \geq u_2(o_1^*)$.
- (c) If $u_1(o_1^*) < u_1(o_1)$, it is also not possible that $u_2(o_1) < u_2(o_1^*)$, as explained in (b). It is also not possible that $u_2(o_1) = u_2(o_1^*)$, because then $u_2(o_1^*) > u_2(o_2)$, and by 4.8, since A is an (w_1, w_2) -maximal and (o_1^*, o_2) is an exchangeable pair in it, $u_1(o_1^*) > u_1(o_2)$. Therefore, $r(o_1^*, o_2) > r(o_1, o_2)$, contradiction.

So in that case, necessarily $u_2(o_1^*) < u_2(o_1)$. Based on that, we now show that $r(o_1, o_2) \leq r(o_1^*, o_1)$.

If also $u_2(o_1^*) > u_2(o_2)$, by $r(o_1, o_2)$ maximality, $r(o_1^*, o_2) \leq r(o_1, o_2)$, and by 4.8, $u_1(o_1) > u_1(o_1^*) > u_1(o_2)$. Then, by B.2 (with $x = o_1, y = o_1^*, z = o_2$), $r(o_1, o_2) \leq r(o_1^*, o_1)$.

Else, $u_2(o_1^*) \leq u_2(o_2)$. Since $u_1(o_2) < u_1(o_1)$ and $u_1(o_1^*) < u_1(o_1)$, there are two options:

- $u_1(o_1^*) < u_1(o_2) < u_1(o_1)$.
By Lemma 4.7 we know that $r(o_1, o_2) \leq \frac{w_1}{w_2}, r(o_1^*, o_2) \geq \frac{w_1}{w_2}$, so $r(o_1, o_2) \leq r(o_1^*, o_2)$, and by B.3 (with $x = o_1, y = o_2, z = o_1^*$), $r(o_1, o_2) \leq r(o_1^*, o_1)$.
- $u_1(o_2) \leq u_1(o_1^*) < u_1(o_1)$.
Then, $u_2(o_1) - u_2(o_2) \leq u_2(o_1) - u_2(o_1^*)$ and $u_1(o_1) - u_1(o_2) \geq u_1(o_1) - u_1(o_1^*)$. So $r(o_1, o_2) = \frac{u_2(o_1) - u_2(o_2)}{u_1(o_1) - u_1(o_2)} \leq \frac{u_2(o_1) - u_2(o_1^*)}{u_1(o_1) - u_1(o_1^*)} = r(o_1^*, o_1)$.

4. Pairs in the form (o_2, o_2^*) , $o_2^* \in A'_2, o_2^* \neq o_1$:

Exactly the same arguments in case 3 (but with o_2, o_2^*), prove this case.

Therefore, by Lemma 4.7 (iii), A' is (w'_1, w'_2) -maximal, and $\frac{w'_1}{w'_2} \leq \frac{w_1}{w_2}$ with $w'_1 + w'_2 = 1$ implies $w'_1 \leq w_1$ and $w'_2 \geq w_2$. \square